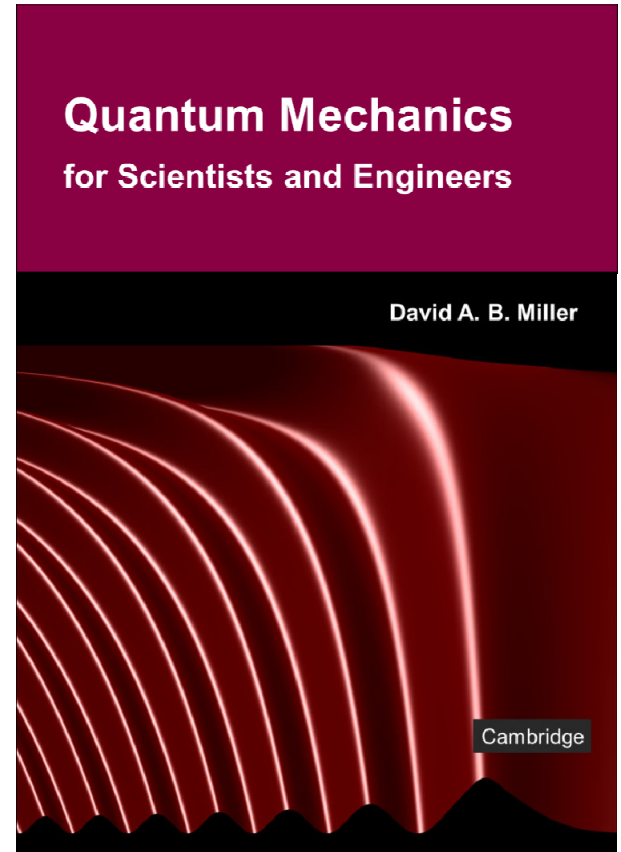


42 Fermion annihilation and creation operators

Slides: Lecture 42a Describing fermion states

Text reference: Quantum Mechanics for Scientists and Engineers

Section Chapter 16 introduction and Section 16.1 up to beginning of "Fermion creation operators"





Fermion annihilation and creation operators

Quantum mechanics for scientists and engineers

David Miller



Fermion annihilation and creation operators



Describing fermion states

Quantum mechanics for scientists and engineers

David Miller

Description and ordering of multiple fermion states

We can write a basis state for multiple identical fermions as

$$\begin{aligned} |\psi_{N; a, b, \dots, n}\rangle &= \frac{1}{\sqrt{N!}} \sum_{\hat{P}=1}^{N!} \pm \hat{P} |1, a\rangle |2, b\rangle |3, c\rangle \dots |N, n\rangle \rangle \\ &\equiv \frac{1}{\sqrt{N!}} \begin{vmatrix} |1, a\rangle & |2, a\rangle & \dots & |N, a\rangle \\ |1, b\rangle & |2, b\rangle & \dots & |N, b\rangle \\ \vdots & \vdots & \ddots & \vdots \\ |1, n\rangle & |2, n\rangle & \dots & |N, n\rangle \end{vmatrix} \end{aligned}$$

Here, there are N identical fermions, and they occupy single-particle basis states a, b, \dots, n

Single-particle fermion states

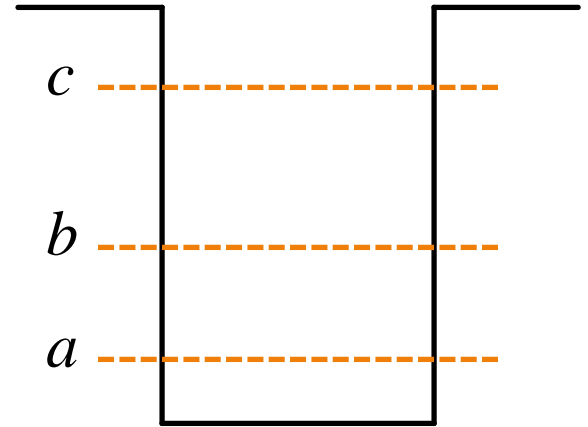
Single-particle basis states are individual states a fermion can occupy

and here each has a lower case letter associated with it

For example

each possible electron state in a potential well or atom

corresponds to a different single-particle basis state here



Multiple fermion basis states

Though $|\psi_{N; a, b, \dots, n}\rangle =$

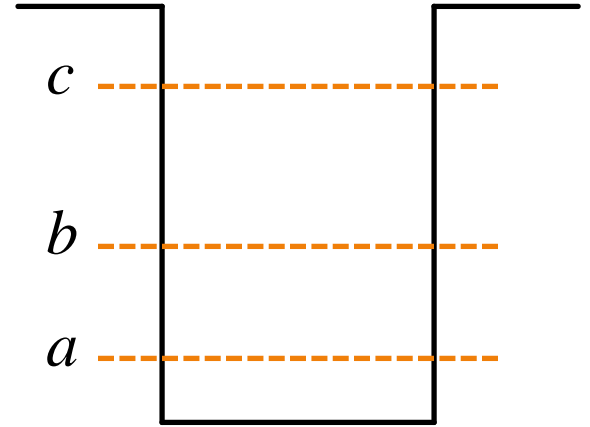
$$\frac{1}{\sqrt{N!}} \sum_{\hat{P}} \pm \hat{P} |1, a\rangle |2, b\rangle |3, c\rangle \dots |N, n\rangle$$

might seem to imply that each of the possible states is occupied

that is not in general the case

Very few of the possible single-particle states are likely occupied

in any given multiple fermion basis state



Multiple fermion basis states

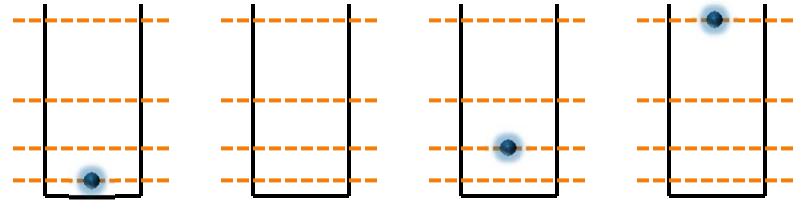
We might have three electrons
in with four potential wells

and be considering a
(multiple particle) basis
state in which there is

1 electron in the ground
state of well 1

1 in the 2nd state of well 3

1 in the 4th state of well 4



Multiple fermion basis states

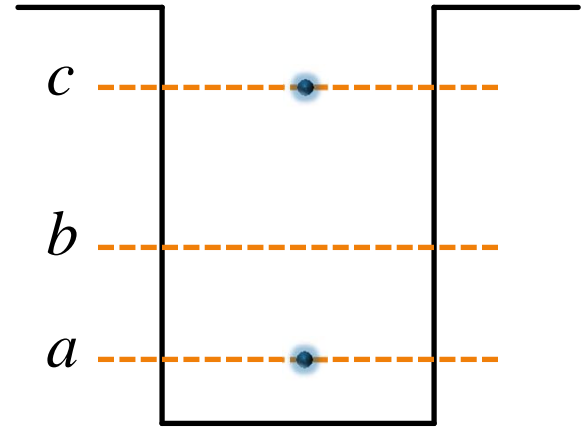
A basis state that might have two electrons in one well

e.g., one on the lowest state and one on the third state

though this is not necessarily an eigenstate of the Hamiltonian

The first electron would repel the second electron

so the second electron would not see a simple square potential



Multiple fermion basis states



For manipulations in the fermion case, we must

- define one standard order of labeling of the single-particle basis states
- in the determinants for the multiple fermion basis functions

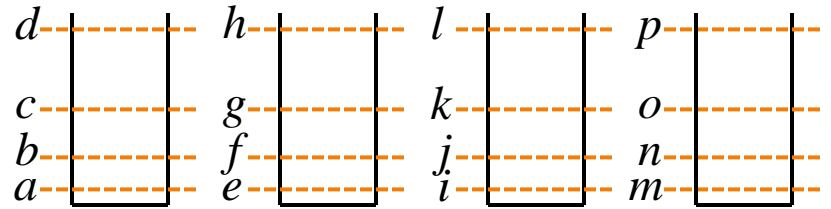
Multiple fermion basis states

For example, if we had a system with four potential wells

we might label sequentially
all of the states in well 1

then next all of the states in
well 2

and so on



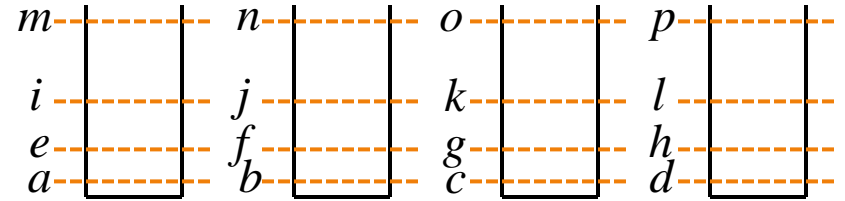
Multiple fermion basis states

We could choose some other labeling sequence

labeling all of the first states
in wells 1 through 4

then all of the second states
in wells 1 through 4

and so on



Multiple fermion basis states

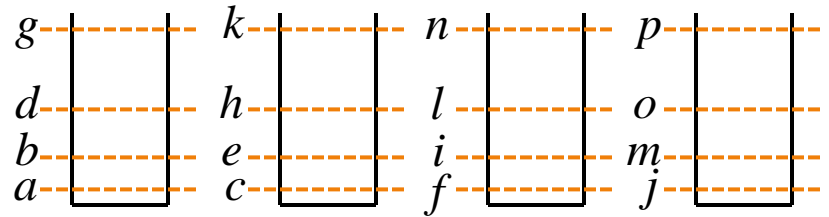
We could choose some other labeling sequence

labeling all of the first states
in wells 1 through 4

then all of the second states
in wells 1 through 4

and so on

or we could even choose some
more complicated labeling
sequence

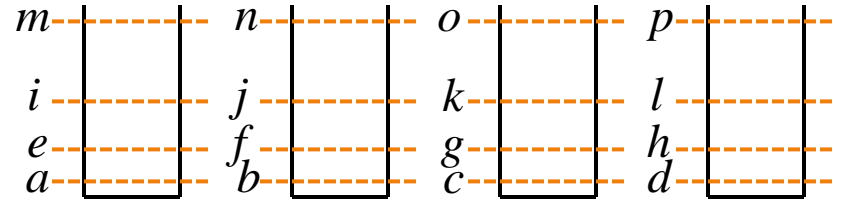


Multiple fermion basis states

It does not matter what
sequence we choose
but we have to have one
standard labeling sequence

Here, we label the single-
particle basis states using the
lower case letters

and those will be in
alphabetical sequence in
our standard order



Occupation number representation

We might, for example, have a basis state corresponding to three identical fermions

one in state b , one in state k , and one in state m

In standard order, we would write that state as

$$|\psi_{3;b,k,m}\rangle = \frac{1}{\sqrt{3!}} \begin{vmatrix} |1,b\rangle & |2,b\rangle & |3,b\rangle \\ |1,k\rangle & |2,k\rangle & |3,k\rangle \\ |1,m\rangle & |2,m\rangle & |3,m\rangle \end{vmatrix} \equiv |0_a, 1_b, 0_c, \dots, 1_k, 0_l, \dots, 1_m, 0_n \dots\rangle$$

Here we have also introduced another notation

the occupation number notation

similar to the boson occupation number notation

Occupation number representation

In this occupation number representation, as in the state

$$|\psi_{3;b,k,m}\rangle = \frac{1}{\sqrt{3!}} \begin{vmatrix} |1,b\rangle & |2,b\rangle & |3,b\rangle \\ |1,k\rangle & |2,k\rangle & |3,k\rangle \\ |1,m\rangle & |2,m\rangle & |3,m\rangle \end{vmatrix} \equiv |0_a, 1_b, 0_c, \dots, 1_k, 0_l, \dots, 1_m, 0_n \dots\rangle$$

0_a in the ket means that the single-particle fermion state (or fermion mode) a is empty, and

1_b means state b is occupied

Because this is a fermion state

the determinant combination of the different fermions to the occupied states is understood

States not in standard order

We could also write a state that was not in standard order for the rows

$$\text{e.g., } |\psi_{3;k,b,m}\rangle = \frac{1}{\sqrt{3!}} \begin{vmatrix} |1,k\rangle & |2,k\rangle & |3,k\rangle \\ |1,b\rangle & |2,b\rangle & |3,b\rangle \\ |1,m\rangle & |2,m\rangle & |3,m\rangle \end{vmatrix}$$

To get that state into standard order for the rows

we would have to swap the first and second rows

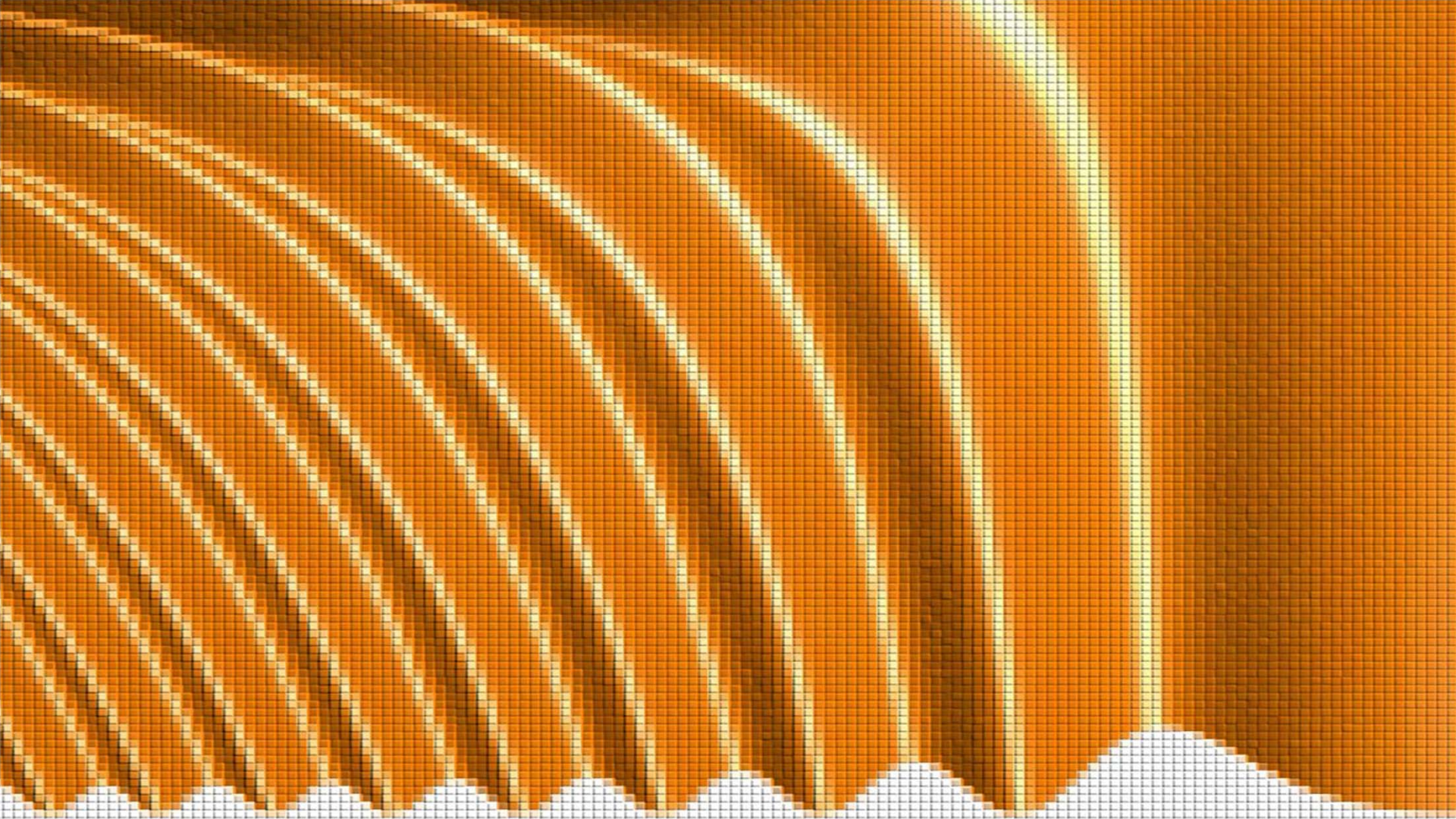
If we swap two adjacent rows in a determinant

we have to multiply the determinant by -1

States not in standard order

So, swapping the top two rows, we have

$$\begin{aligned} |\psi_{3;k,b,m}\rangle &= -\frac{1}{\sqrt{3!}} \begin{vmatrix} |1,k\rangle & |2,k\rangle & |3,k\rangle \\ |1,b\rangle & |2,b\rangle & |3,b\rangle \\ |1,m\rangle & |2,m\rangle & |3,m\rangle \end{vmatrix} \\ &= -|0_a, 1_b, 0_c, \dots, 1_k, 0_l, \dots, 1_m, 0_n \dots\rangle \\ &= -|\psi_{3;b,k,m}\rangle \end{aligned}$$

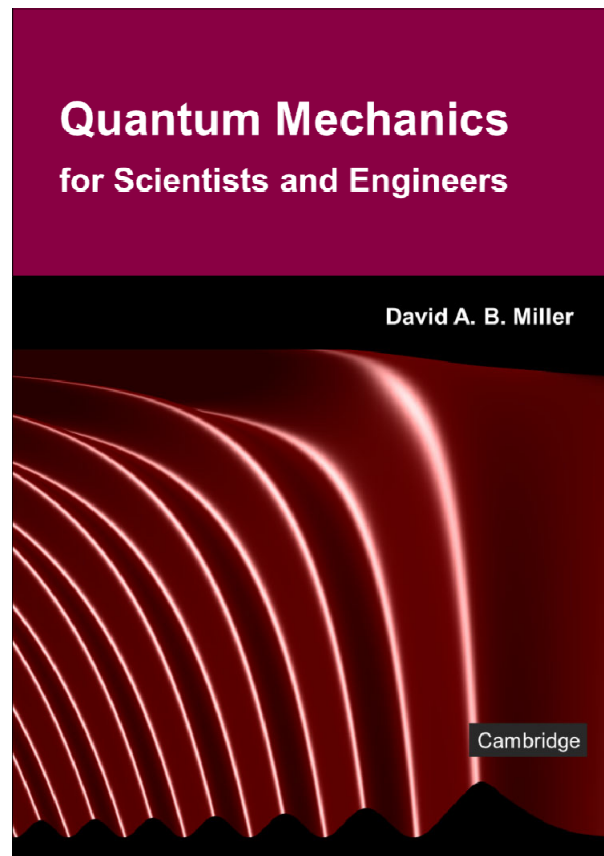


42 Fermion annihilation and creation operators

Slides: Lecture 42b Creation operators

Text reference: Quantum Mechanics for Scientists and Engineers

Section 16.1 subsection "Fermion creation operators"





Fermion annihilation and creation operators



Creation operators



Quantum mechanics for scientists and engineers



David Miller

Fermion creation operators

Now we postulate

a fermion creation operator

for fermion “mode” or single-particle
basis state k

and write it as \hat{b}_k^\dagger

It must take any state in which single-particle
basis state k is empty

and turn it into one

in which this state k is occupied

Constructing the creation operator

Suppose we start with the state where

single-particle states b and m are occupied

but state k and all other states are not

In the permutation notation, we therefore propose that

\hat{b}_k^\dagger has the following effect on that state

$$\hat{b}_k^\dagger \frac{1}{\sqrt{2!}} \sum_{\hat{P}=1}^{2!} \pm \hat{P} ||1, b\rangle |2, m\rangle\rangle = \frac{1}{\sqrt{3!}} \sum_{\hat{P}=1}^{3!} \pm \hat{P} ||1, b\rangle |2, m\rangle |3, k\rangle\rangle$$

So, \hat{b}_k^\dagger adds a third particle into the system

and we propose that it adds it to the end of the list

Constructing the creation operator

Adding to the end of the list is equivalent to

adding a row to the bottom of the determinant (and a column to the right)

i.e., now dropping the normalization factors for convenience but keeping the sign behavior

$$\hat{b}_k^\dagger \begin{vmatrix} |1,b\rangle & |2,b\rangle \\ |1,m\rangle & |2,m\rangle \end{vmatrix} \propto \begin{vmatrix} |1,b\rangle & |2,b\rangle & |3,b\rangle \\ |1,m\rangle & |2,m\rangle & |3,m\rangle \\ |1,k\rangle & |2,k\rangle & |3,k\rangle \end{vmatrix}$$

Note the sequence in the permutation notation is the same as the sequence down this leading diagonal

Constructing the creation operator

For this case

$$\hat{b}_k^\dagger \begin{vmatrix} |1,b\rangle & |2,b\rangle \\ |1,m\rangle & |2,m\rangle \end{vmatrix} \propto \begin{vmatrix} |1,b\rangle & |2,b\rangle & |3,b\rangle \\ |1,m\rangle & |2,m\rangle & |3,m\rangle \\ |1,k\rangle & |2,k\rangle & |3,k\rangle \end{vmatrix} = - \begin{vmatrix} |1,b\rangle & |2,b\rangle & |3,b\rangle \\ |1,k\rangle & |2,k\rangle & |3,k\rangle \\ |1,m\rangle & |2,m\rangle & |3,m\rangle \end{vmatrix}$$

the determinant is not written in standard order

To get this particular determinant into standard order

we need to swap the bottom two rows

and in performing this one swap

we must therefore multiply the determinant by -1

Constructing the creation operator

Suppose now that we add another particle

this time in state j

using the operator \hat{b}_j^\dagger

Then we have

$$\hat{b}_j^\dagger \hat{b}_k^\dagger \begin{vmatrix} |1, b\rangle & |2, b\rangle \\ |1, m\rangle & |2, m\rangle \end{vmatrix} \propto -\hat{b}_j^\dagger \begin{vmatrix} |1, b\rangle & |2, b\rangle & |3, b\rangle \\ |1, k\rangle & |2, k\rangle & |3, k\rangle \\ |1, m\rangle & |2, m\rangle & |3, m\rangle \end{vmatrix}$$

Constructing the creation operator

Suppose now that we add another particle

this time in state j

using the operator \hat{b}_j^\dagger

Then we have

$$\hat{b}_j^\dagger \hat{b}_k^\dagger \begin{vmatrix} |1, b\rangle & |2, b\rangle \\ |1, m\rangle & |2, m\rangle \end{vmatrix} \propto \begin{vmatrix} |1, b\rangle & |2, b\rangle & |3, b\rangle & |4, b\rangle \\ |1, k\rangle & |2, k\rangle & |3, k\rangle & |4, k\rangle \\ |1, m\rangle & |2, m\rangle & |3, m\rangle & |4, m\rangle \\ |1, j\rangle & |2, j\rangle & |3, j\rangle & |4, j\rangle \end{vmatrix}$$

Constructing the creation operator

To get to standard order

we have to swap the bottom j row with the adjacent m row

multiplying by -1

$$\hat{b}_j^\dagger \hat{b}_k^\dagger \begin{vmatrix} |1,b\rangle & |2,b\rangle \\ |1,m\rangle & |2,m\rangle \end{vmatrix} \propto (-1) \begin{vmatrix} |1,b\rangle & |2,b\rangle & |3,b\rangle & |4,b\rangle \\ |1,k\rangle & |2,k\rangle & |3,k\rangle & |4,k\rangle \\ |1,m\rangle & |2,m\rangle & |3,m\rangle & |4,m\rangle \\ |1,j\rangle & |2,j\rangle & |3,j\rangle & |4,j\rangle \end{vmatrix}$$

Constructing the creation operator

Then we swap the j row

now second from the bottom

with the adjacent k row,

multiplying again by -1

$$\hat{b}_j^\dagger \hat{b}_k^\dagger \begin{vmatrix} |1,b\rangle & |2,b\rangle \\ |1,m\rangle & |2,m\rangle \end{vmatrix} \propto - (-1)^2 \begin{vmatrix} |1,b\rangle & |2,b\rangle & |3,b\rangle & |4,b\rangle \\ |1,k\rangle & |2,k\rangle & |3,k\rangle & |4,k\rangle \\ |1,j\rangle & |2,j\rangle & |3,j\rangle & |4,j\rangle \\ |1,m\rangle & |2,m\rangle & |3,m\rangle & |4,m\rangle \end{vmatrix}$$

Constructing the creation operator

So finally

$$\hat{b}_j^\dagger \hat{b}_k^\dagger \begin{vmatrix} |1,b\rangle & |2,b\rangle \\ |1,m\rangle & |2,m\rangle \end{vmatrix} \propto (-1)^2 \begin{vmatrix} |1,b\rangle & |2,b\rangle & |3,b\rangle & |4,b\rangle \\ |1,j\rangle & |2,j\rangle & |3,j\rangle & |4,j\rangle \\ |1,k\rangle & |2,k\rangle & |3,k\rangle & |4,k\rangle \\ |1,m\rangle & |2,m\rangle & |3,m\rangle & |4,m\rangle \end{vmatrix}$$

Constructing the creation operator

Now suppose we do this two-particle creation operation in the opposite order

First, similarly to before, but first with \hat{b}_j^\dagger

and performing the necessary swap of the bottom two rows

$$\hat{b}_j^\dagger \begin{vmatrix} |1, b\rangle & |2, b\rangle \\ |1, m\rangle & |2, m\rangle \end{vmatrix} \propto - \begin{vmatrix} |1, b\rangle & |2, b\rangle & |3, b\rangle \\ |1, j\rangle & |2, j\rangle & |3, j\rangle \\ |1, m\rangle & |2, m\rangle & |3, m\rangle \end{vmatrix}$$

Constructing the creation operator

Next, if we operate with \hat{b}_k^\dagger

$$\hat{b}_k^\dagger \hat{b}_j^\dagger \begin{vmatrix} |1, b\rangle & |2, b\rangle \\ |1, m\rangle & |2, m\rangle \end{vmatrix} \propto -\hat{b}_k^\dagger \begin{vmatrix} |1, b\rangle & |2, b\rangle & |3, b\rangle \\ |1, j\rangle & |2, j\rangle & |3, j\rangle \\ |1, m\rangle & |2, m\rangle & |3, m\rangle \end{vmatrix}$$

Constructing the creation operator

Next, if we operate with \hat{b}_k^\dagger

adding a k row to the bottom (and a column to the right)

we obtain

$$\hat{b}_k^\dagger \hat{b}_j^\dagger \begin{vmatrix} |1,b\rangle & |2,b\rangle \\ |1,m\rangle & |2,m\rangle \end{vmatrix} \propto \begin{vmatrix} |1,b\rangle & |2,b\rangle & |3,b\rangle & |4,b\rangle \\ |1,j\rangle & |2,j\rangle & |3,j\rangle & |4,j\rangle \\ |1,m\rangle & |2,m\rangle & |3,m\rangle & |4,m\rangle \\ |1,k\rangle & |2,k\rangle & |3,k\rangle & |4,k\rangle \end{vmatrix}$$

Constructing the creation operator

Now, however

we only have to swap adjacent rows once
not twice

to get the determinant into standard order

$$\hat{b}_k^\dagger \hat{b}_j^\dagger \begin{vmatrix} |1,b\rangle & |2,b\rangle \\ |1,m\rangle & |2,m\rangle \end{vmatrix} \propto + \begin{vmatrix} |1,b\rangle & |2,b\rangle & |3,b\rangle & |4,b\rangle \\ |1,j\rangle & |2,j\rangle & |3,j\rangle & |4,j\rangle \\ |1,m\rangle & |2,m\rangle & |3,m\rangle & |4,m\rangle \\ |1,k\rangle & |2,k\rangle & |3,k\rangle & |4,k\rangle \end{vmatrix}$$

This result is -1 times the result from that of
the operators in the order $\hat{b}_j^\dagger \hat{b}_k^\dagger$

Sign behavior for creation operator pairs

For example, we would get the same difference in sign

if we had considered the pairs of operators

$$\hat{b}_a^\dagger \hat{b}_k^\dagger \text{ and } \hat{b}_k^\dagger \hat{b}_a^\dagger$$

or the pairs $\hat{b}_j^\dagger \hat{b}_n^\dagger$ and $\hat{b}_n^\dagger \hat{b}_j^\dagger$

Note one of the pairs of operators always results in one more swap of adjacent rows than the other

because it encounters one more row to be swapped

Sign behavior for creation operator pairs

Hence we have the result

valid for any state with single-particle
states j and k initially empty

$$\hat{b}_j^\dagger \hat{b}_k^\dagger + \hat{b}_k^\dagger \hat{b}_j^\dagger = 0$$

In fact

this relation is universally true for any state
as we can now show

Sign behavior for creation operator pairs

For any state in which state k is initially occupied

the fermion creation operator for that state

must have the property that $\hat{b}_k^\dagger |\dots, 1_k, \dots\rangle = 0$

because we cannot create two fermions in one single-particle state

Hence when the single-particle state k is occupied

trivially we have $\hat{b}_j^\dagger \hat{b}_k^\dagger |\dots, 1_k, \dots\rangle = 0$ and $\hat{b}_k^\dagger \hat{b}_j^\dagger |\dots, 1_k, \dots\rangle = 0$

Hence $\hat{b}_j^\dagger \hat{b}_k^\dagger + \hat{b}_k^\dagger \hat{b}_j^\dagger = 0$ still works here

because each individual term is zero

and similarly when state j is initially occupied

Sign behavior for creation operator pairs

We also trivially get the same result $\hat{b}_j^\dagger \hat{b}_k^\dagger + \hat{b}_k^\dagger \hat{b}_j^\dagger = 0$

for any initial state with $j = k$

because we are trying to create at least two fermions in the single-particle state

three if it is already occupied

and so we also get zero for both terms

Hence we conclude that

$$\hat{b}_j^\dagger \hat{b}_k^\dagger + \hat{b}_k^\dagger \hat{b}_j^\dagger = 0$$

is valid for any starting state

Anticommutation relation

A relation of the form $\hat{b}_j^\dagger \hat{b}_k^\dagger + \hat{b}_k^\dagger \hat{b}_j^\dagger = 0$

is called an anticommutation relation

It is like a commutation relation between operators
but with a plus sign in the middle, rather than
the minus sign of a commutation relation

A notation sometimes used for an anticommutator of
two operators is

$$\hat{b}_j^\dagger \hat{b}_k^\dagger + \hat{b}_k^\dagger \hat{b}_j^\dagger \equiv \left\{ \hat{b}_j^\dagger, \hat{b}_k^\dagger \right\}$$

Now we will progressively develop a family of
anticommutation relations for the fermion operators

Formalization of creation operator sign behavior

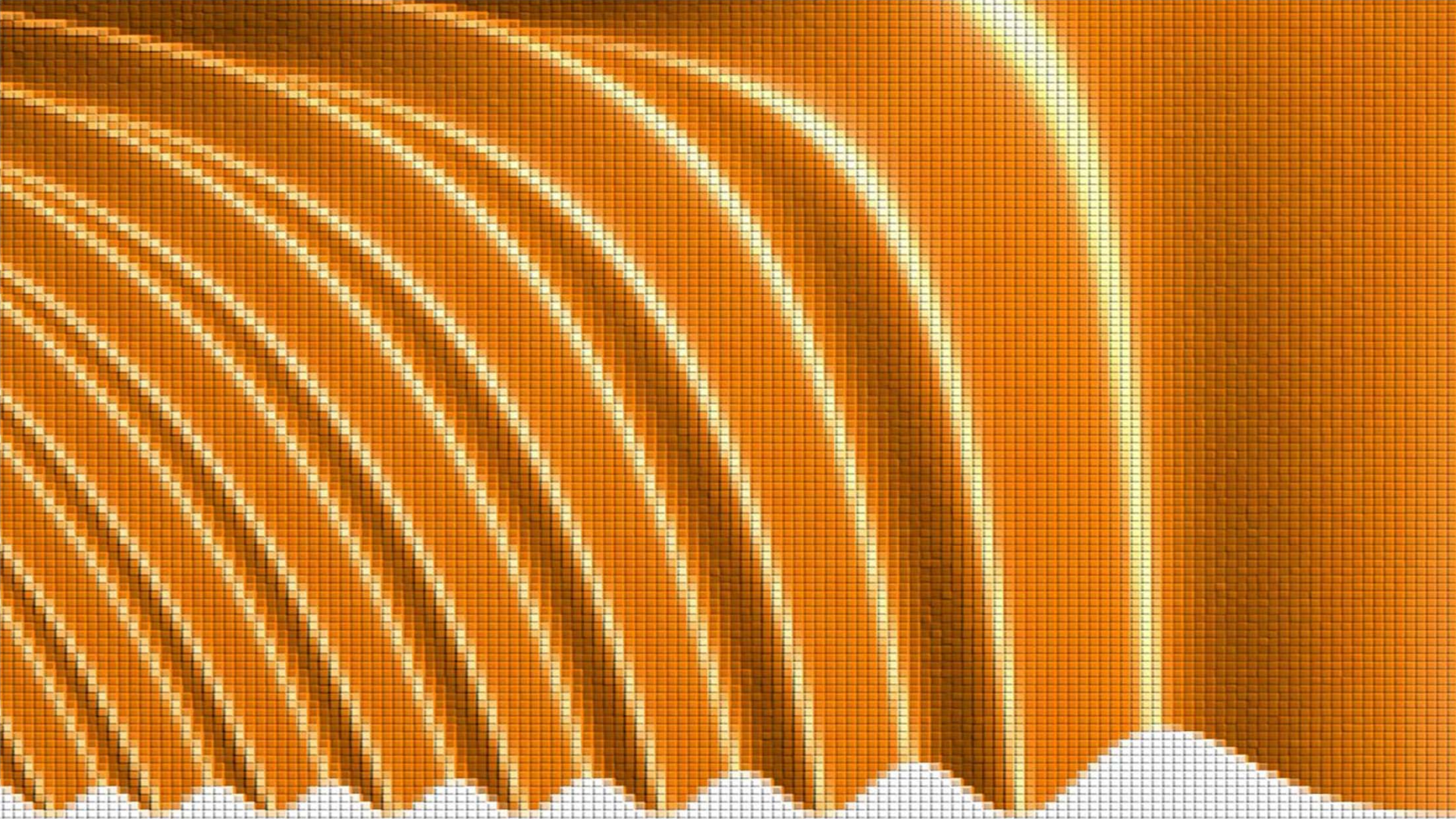
We see, with our choice that we add the particle in state k initially to the end of the list

or, equivalently, to the bottom of the determinant
and then swap it into place

that the number of swaps we have to perform is
the number, S_k , of occupied states that are
after the state k in the standard order

With this definition

we have formally $\hat{b}_k^\dagger |\dots, 0_k, \dots\rangle = (-1)^{S_k} |\dots, 1_k, \dots\rangle$

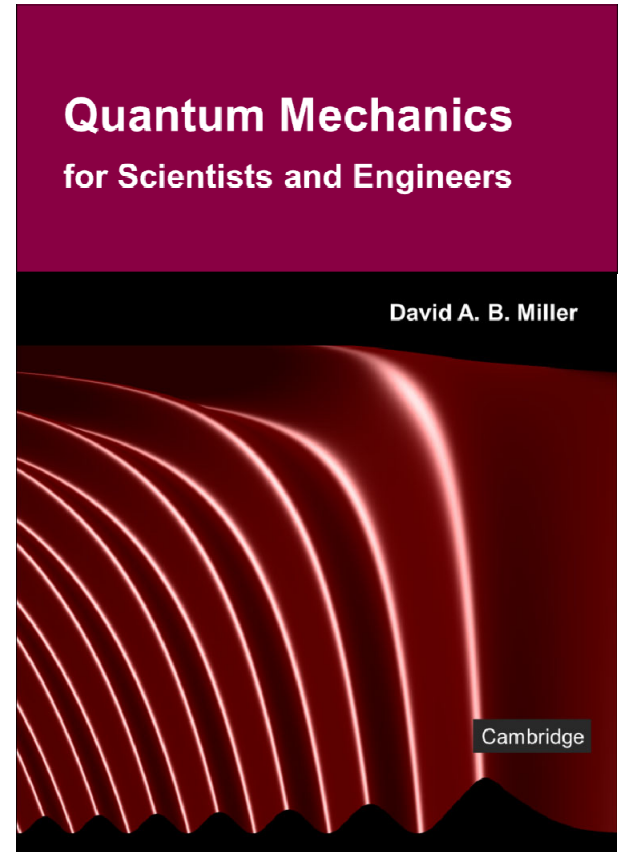


42 Fermion annihilation and creation operators

Slides: Lecture 42c Annihilation operators

Text reference: Quantum Mechanics for Scientists and Engineers

Section 16.1 subsection "Fermion annihilation operators"





Fermion annihilation and creation
operators



Annihilation operators

Quantum mechanics for scientists and engineers

David Miller

Fermion annihilation operators

Now we can proceed to define annihilation operators

From $\hat{b}_k^\dagger |\dots, 0_k, \dots\rangle = (-1)^{S_k} |\dots, 1_k, \dots\rangle$ we can see that

$$\langle \dots, 1_k, \dots | \hat{b}_k^\dagger |\dots, 0_k, \dots\rangle = (-1)^{S_k} \langle \dots, 1_k, \dots | \dots, 1_k, \dots\rangle = (-1)^{S_k}$$

Taking the complex conjugate

or, actually, the Hermitian adjoint, of both sides

$$\begin{aligned} \langle \dots, 1_k, \dots | \hat{b}_k^\dagger |\dots, 0_k, \dots\rangle^\dagger &= \left(\hat{b}_k^\dagger |\dots, 0_k, \dots\rangle \right)^\dagger \left(\langle \dots, 1_k, \dots | \right)^\dagger \\ &= \langle \dots, 0_k, \dots | \hat{b}_k |\dots, 1_k, \dots\rangle = (-1)^{S_k} \end{aligned}$$

So, we deduce $\hat{b}_k |\dots, 1_k, \dots\rangle = (-1)^{S_k} |\dots, 0_k, \dots\rangle$

Fermion annihilation operators

Hence

whereas \hat{b}_k^\dagger creates a fermion in single-particle state k

provided that state was empty

we see from $\hat{b}_k |\dots, 1_k, \dots\rangle = (-1)^{S_k} |\dots, 0_k, \dots\rangle$

\hat{b}_k annihilates a fermion in single-particle state k

provided that state was full

and is called the fermion annihilation operator for state k

Annihilation operator acting on a state

The annihilation operator acting on the Slater determinant progressively

swaps the row corresponding to state k in the determinant with the one below it

until that row gets to the bottom of the determinant

in which case we remove it (and the last column) of the determinant

in an inverse fashion to the process with the creation operator

Anticommutator for annihilation operators

Using the relation (analogous to $\hat{b}_k^\dagger |\dots, 1_k, \dots\rangle = 0$)

$$\hat{b}_k |\dots, 0_k, \dots\rangle = 0$$

which merely states that, if the single-particle state k is empty to start with

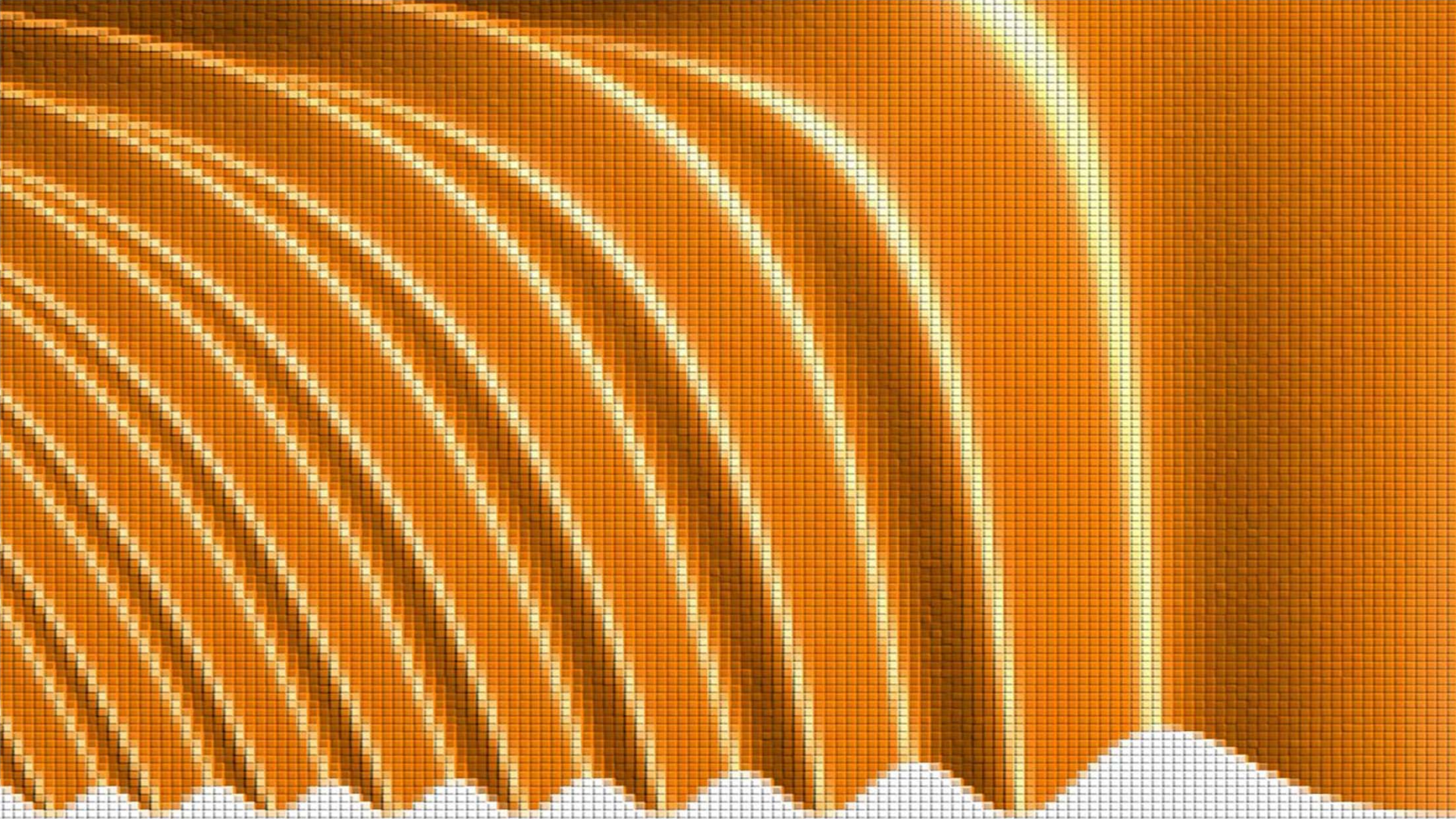
we cannot annihilate another particle from that state

we can argue similarly that

$$\hat{b}_j \hat{b}_k + \hat{b}_k \hat{b}_j = 0$$

which is the annihilation operator anticommutation relation

valid for all states and for $j = k$

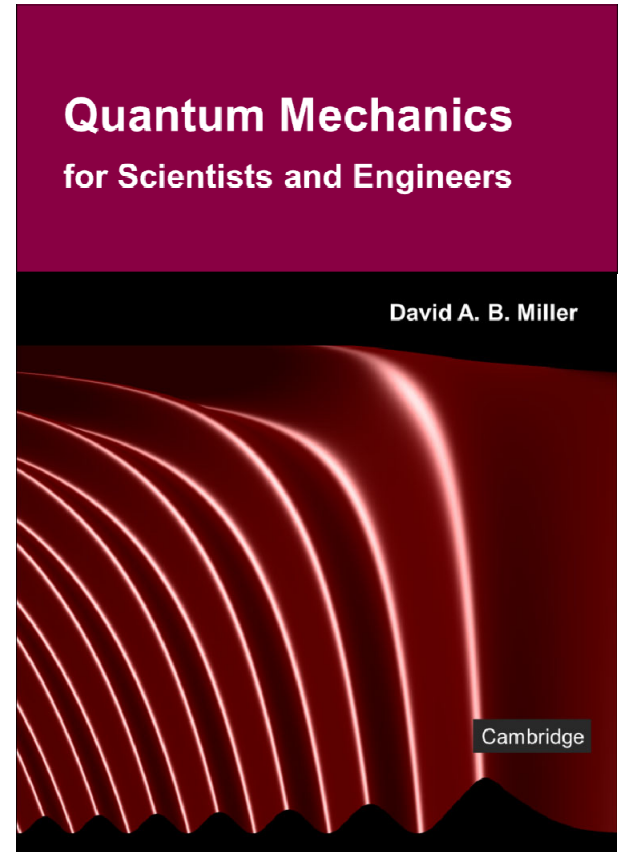


42 Fermion annihilation and creation operators

Slides: Lecture 42d Mixtures of creation and annihilation operators

Text reference: Quantum Mechanics for Scientists and Engineers

Section 16.1 subsection "Mixtures of creation and annihilation operators"





Fermion annihilation and creation operators



Mixtures of creation and annihilation operators

Quantum mechanics for scientists and engineers

David Miller

Mixtures of creation and annihilation operators

Suppose single-particle states b, j , and m are initially occupied

and we operate on this state first with the annihilation operator \hat{b}_j

Then we have

$$\hat{b}_j \begin{vmatrix} |1, b\rangle & |2, b\rangle & |3, b\rangle \\ |1, j\rangle & |2, j\rangle & |3, j\rangle \\ |1, m\rangle & |2, m\rangle & |3, m\rangle \end{vmatrix} = - \begin{vmatrix} |1, b\rangle & |2, b\rangle \\ |1, m\rangle & |2, m\rangle \end{vmatrix}$$

where we had to swap the j and m rows to get the j row to the bottom

Mixtures of creation and annihilation operators

Now we operate with \hat{b}_k^\dagger , obtaining

$$\hat{b}_k^\dagger \hat{b}_j \begin{vmatrix} |1, b\rangle & |2, b\rangle & |3, b\rangle \\ |1, j\rangle & |2, j\rangle & |3, j\rangle \\ |1, m\rangle & |2, m\rangle & |3, m\rangle \end{vmatrix} = -\hat{b}_k^\dagger \begin{vmatrix} |1, b\rangle & |2, b\rangle \\ |1, m\rangle & |2, m\rangle \end{vmatrix} = \begin{vmatrix} |1, b\rangle & |2, b\rangle & |3, b\rangle \\ |1, k\rangle & |2, k\rangle & |3, k\rangle \\ |1, m\rangle & |2, m\rangle & |3, m\rangle \end{vmatrix}$$

where the minus sign is cancelled because we had to swap the k row from the bottom with the m row

Mixtures of creation and annihilation operators

Next let us consider applying these operators in the opposite order, starting now with \hat{b}_k^\dagger

$$\hat{b}_k^\dagger \begin{vmatrix} |1, b\rangle & |2, b\rangle & |3, b\rangle \\ |1, j\rangle & |2, j\rangle & |3, j\rangle \\ |1, m\rangle & |2, m\rangle & |3, m\rangle \end{vmatrix} = - \begin{vmatrix} |1, b\rangle & |2, b\rangle & |3, b\rangle & |4, b\rangle \\ |1, j\rangle & |2, j\rangle & |3, j\rangle & |4, j\rangle \\ |1, k\rangle & |2, k\rangle & |3, k\rangle & |4, k\rangle \\ |1, m\rangle & |2, m\rangle & |3, m\rangle & |4, m\rangle \end{vmatrix}$$

where we had to swap the k row from the bottom with the m row

Mixtures of creation and annihilation operators

Applying the \hat{b}_j operator now gives

$$\hat{b}_j \hat{b}_k^\dagger \begin{vmatrix} |1,b\rangle & |2,b\rangle & |3,b\rangle \\ |1,j\rangle & |2,j\rangle & |3,j\rangle \\ |1,m\rangle & |2,m\rangle & |3,m\rangle \end{vmatrix} = -\hat{b}_j \begin{vmatrix} |1,b\rangle & |2,b\rangle & |3,b\rangle & |4,b\rangle \\ |1,j\rangle & |2,j\rangle & |3,j\rangle & |4,j\rangle \\ |1,k\rangle & |2,k\rangle & |3,k\rangle & |4,k\rangle \\ |1,m\rangle & |2,m\rangle & |3,m\rangle & |4,m\rangle \end{vmatrix}$$

$$= - \begin{vmatrix} |1,b\rangle & |2,b\rangle & |3,b\rangle \\ |1,k\rangle & |2,k\rangle & |3,k\rangle \\ |1,m\rangle & |2,m\rangle & |3,m\rangle \end{vmatrix}$$

In operating with \hat{b}_j , two swaps are required because we have to swap past both the m and k rows.

Mixtures of creation and annihilation operators

As before, we find an additional row swap required with one order of operators rather than the other

The result

$$\hat{b}_k^\dagger \hat{b}_j \begin{vmatrix} |1,b\rangle & |2,b\rangle & |3,b\rangle \\ |1,j\rangle & |2,j\rangle & |3,j\rangle \\ |1,m\rangle & |2,m\rangle & |3,m\rangle \end{vmatrix} = \begin{vmatrix} |1,b\rangle & |2,b\rangle & |3,b\rangle \\ |1,k\rangle & |2,k\rangle & |3,k\rangle \\ |1,m\rangle & |2,m\rangle & |3,m\rangle \end{vmatrix}$$

is minus the result

$$\hat{b}_j \hat{b}_k^\dagger \begin{vmatrix} |1,b\rangle & |2,b\rangle & |3,b\rangle \\ |1,j\rangle & |2,j\rangle & |3,j\rangle \\ |1,m\rangle & |2,m\rangle & |3,m\rangle \end{vmatrix} = - \begin{vmatrix} |1,b\rangle & |2,b\rangle & |3,b\rangle \\ |1,k\rangle & |2,k\rangle & |3,k\rangle \\ |1,m\rangle & |2,m\rangle & |3,m\rangle \end{vmatrix}$$

Mixtures of creation and annihilation operators

Hence, at least when operating on states when single-particle state j is initially full and single-particle state k is initially empty

$$\hat{b}_j \hat{b}_k^\dagger + \hat{b}_k^\dagger \hat{b}_j = 0$$

Again, if state j is initially empty, then both pairs of operators will lead to a zero result

and similarly if state k is initially full

Hence, as long as states j and k are different states
this relation is universally true

Mixtures of creation and annihilation operators

The only special case we have to consider more carefully here is for $j = k$

Suppose single-particle state k is initially full

Then we have

$$\hat{b}_k \hat{b}_k^\dagger \begin{vmatrix} |1, b\rangle & |2, b\rangle & |3, b\rangle \\ |1, k\rangle & |2, k\rangle & |3, k\rangle \\ |1, m\rangle & |2, m\rangle & |3, m\rangle \end{vmatrix} = 0$$

because \hat{b}_k^\dagger operating on this state gives zero

Mixtures of creation and annihilation operators

For the other order of operators, we have

$$\hat{b}_k^\dagger \hat{b}_k \begin{vmatrix} |1,b\rangle & |2,b\rangle & |3,b\rangle \\ |1,k\rangle & |2,k\rangle & |3,k\rangle \\ |1,m\rangle & |2,m\rangle & |3,m\rangle \end{vmatrix} = -\hat{b}_k^\dagger \begin{vmatrix} |1,b\rangle & |2,b\rangle \\ |1,m\rangle & |1,m\rangle \end{vmatrix} = \begin{vmatrix} |1,b\rangle & |2,b\rangle & |3,b\rangle \\ |1,k\rangle & |2,k\rangle & |3,k\rangle \\ |1,m\rangle & |2,m\rangle & |3,m\rangle \end{vmatrix}$$

It is left as an exercise to repeat this derivation for the situation where state k is initially empty

In both cases, the result is the same

One or other of the pairs returns the original state
and the other pair returns zero

Anticommutation relation for mixed operators

Hence we can say that $\hat{b}_k \hat{b}_k^\dagger + \hat{b}_k^\dagger \hat{b}_k = 1$

Putting this together with $\hat{b}_j \hat{b}_k^\dagger + \hat{b}_k^\dagger \hat{b}_j = 0$

we can write the anticommutation relation for mixed annihilation and creation operators

$$\hat{b}_j \hat{b}_k^\dagger + \hat{b}_k^\dagger \hat{b}_j = \delta_{jk}$$

Fermion number operator

Finally, we note that $\hat{b}_k^\dagger \hat{b}_k$

is the fermion number operator for the state k

i.e., it will tell us the number of fermions occupying state k

If the state is initially empty

it will return the value zero

and if the state is initially full

it will return the value 1

We can write this as

$$\hat{N}_k = \hat{b}_k^\dagger \hat{b}_k$$

