Finding and counting channels with waves

stanford.io/3WSnn0S

David Miller
Stanford University

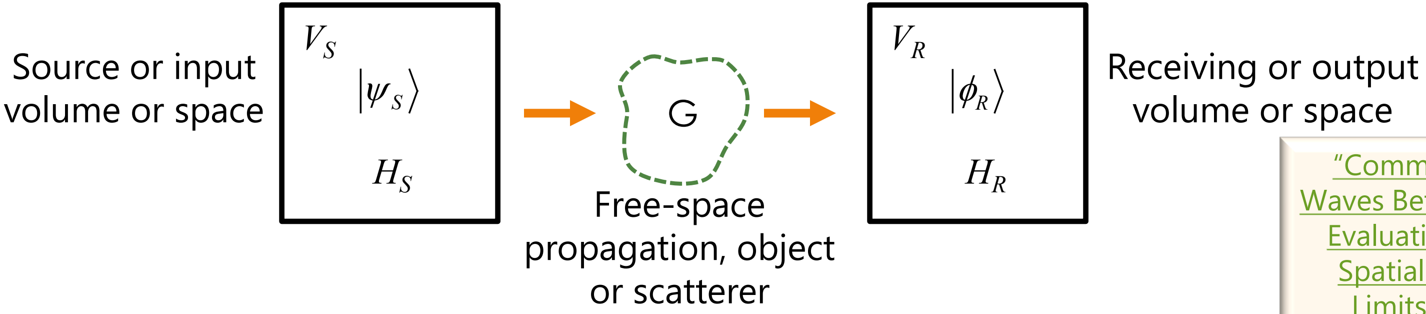
Finding and counting channels with waves

We need to count and find the spatial channels with waves from “sources” to “receivers”

- For optical communications
- For wireless communications
- For sensing
- For thermal physics
- For quantum physics
- For designing nanophotonic and metasurface structures
- For understanding what we mean by “diffraction” when we are working with “volumes”, not just “surfaces”

stanford.io/3WSnn0S

Communication modes



To understand orthogonal channels in optics

think of the mapping from source space to receiving space

as some linear operator or Green's function G

Then we can use singular value decomposition (SVD)

to find the orthogonal channels

"**communication modes**"

the orthogonal source functions that couple one by one

to orthogonal received waves

"Communicating with Waves Between Volumes – Evaluating Orthogonal Spatial Channels and Limits on Coupling Strengths," Appl. Opt. **39**, 1681 (2000).

"All linear optical devices are mode converters," Opt. Express **20**, 23985-23993 (2012)

"Waves, modes, communications and optics," Adv. Opt. Photon. 11, 679 (2019)

Finding and counting channels with waves

More generally

- as a fundamental optical question
  - How many ***usable*** channels are there between some source volume and some receiving volume?
- This is a non-trivial question
  - because the sets of functions themselves may be mathematically infinite
- Does this question have a meaningful and general answer
  - and some clear physical insight?
  - **Yes!**

How many waves can get out of a volume?

stanford.io/3WSnn0S

David Miller, *Stanford University*
Zeyu Kuang, Owen Miller, *Yale University*

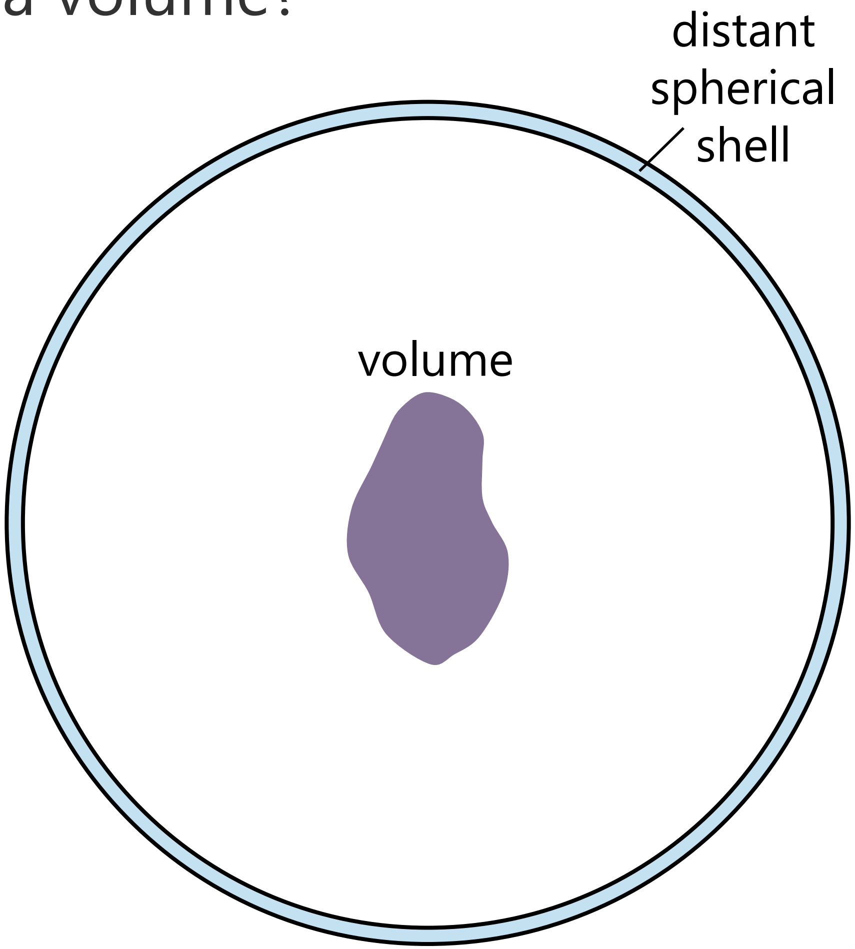
How many waves can get out of a volume?

Suppose we have some arbitrary volume

- which could contain
  - some optical source
    - or some set of antenna elements

Can we deduce just how many waves or channels

- can effectively get out of it
  - propagating into the far field
    - e.g., to a distant spherical shell?



The rigorous approach to channels between volumes

We return to the the singular-value decomposition of the coupling operator G_{SR}

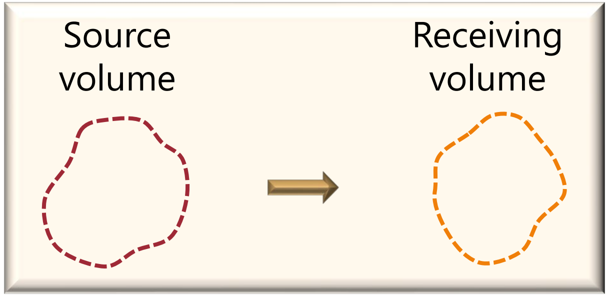
giving orthogonal source functions $|\psi_{Sj}\rangle$

that couple, one by one, to orthogonal received waves $|\phi_{Rj}\rangle$

with some coupling strength s_j

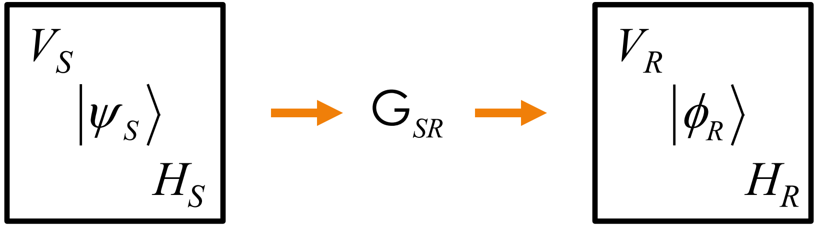
These pairs of functions $|\psi_{Sj}\rangle$ and $|\phi_{Rj}\rangle$

are the "communication modes"



"Waves, modes, communications and optics," Adv. Opt. Photon. 11, 679-825 (2019)

Source or input volume V_S or space H_S

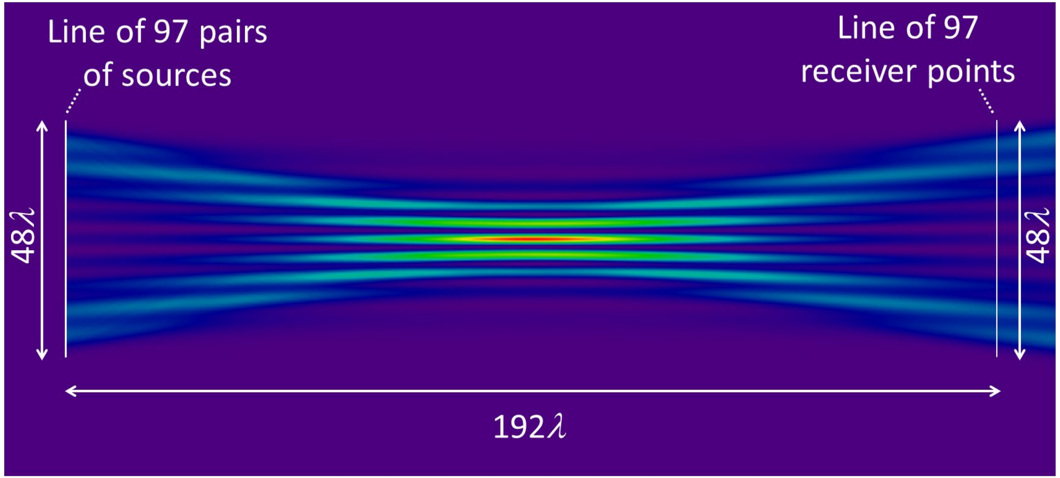


Receiving or output volume V_R or space H_R

A paraxial example

Suppose we have a line of sources and a line of receiver points

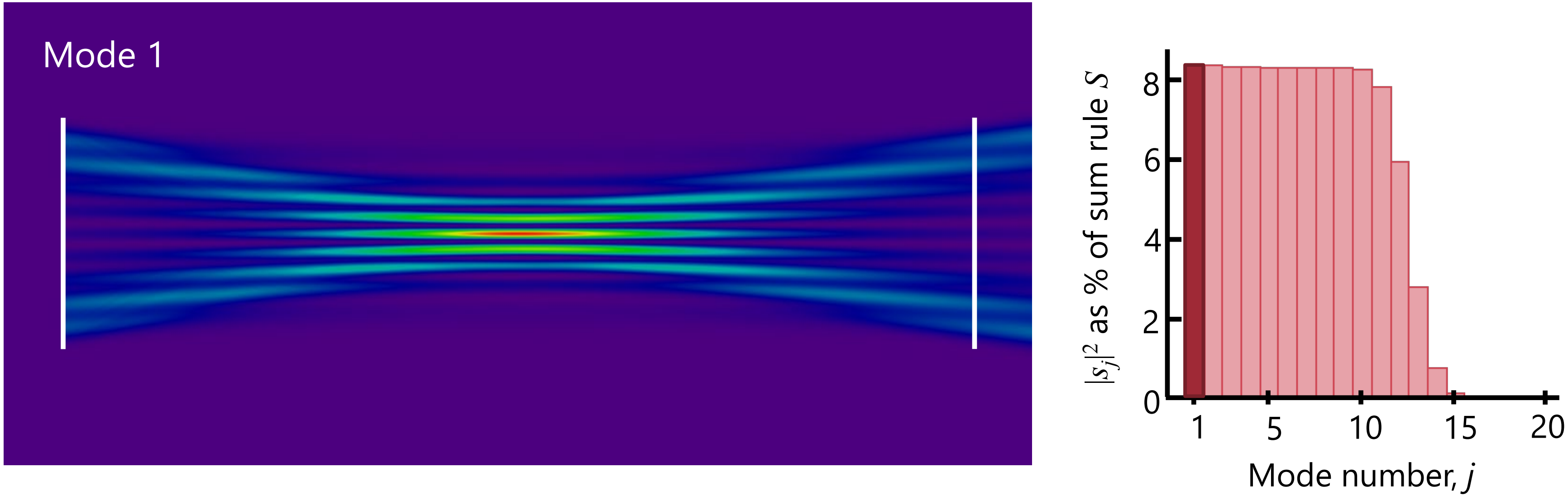
here in an approximately "paraxial" set of dimensions



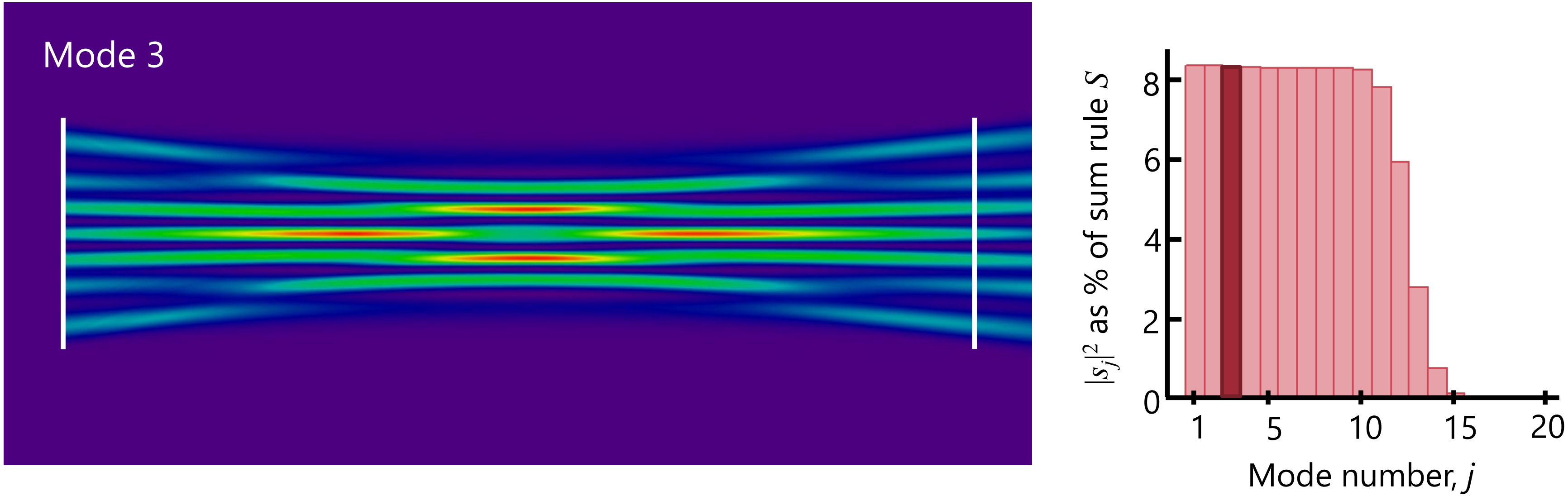
and we establish the communication modes between them

The picture shows the cross-section of the intensity in the plane

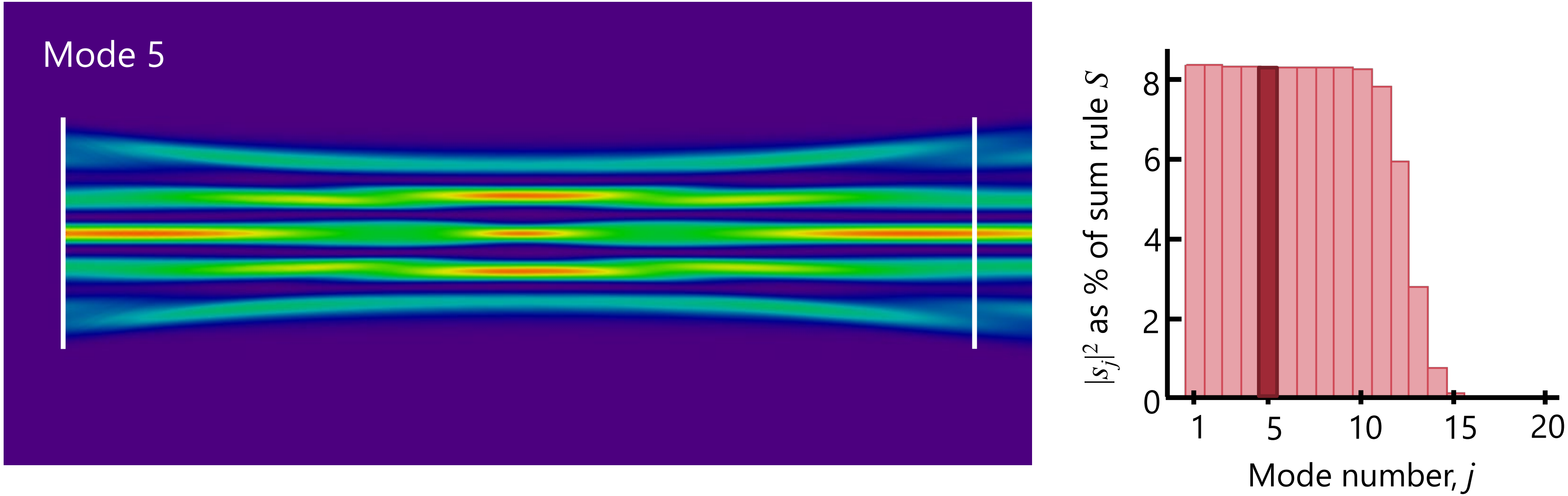
here for the most strongly coupled mode

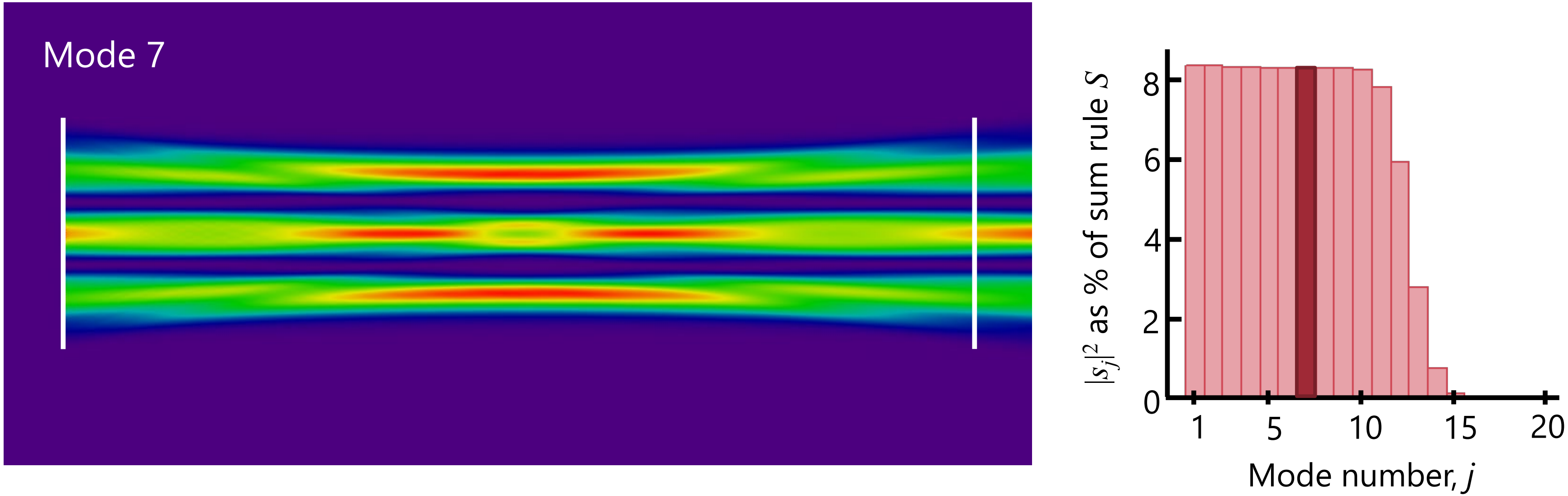


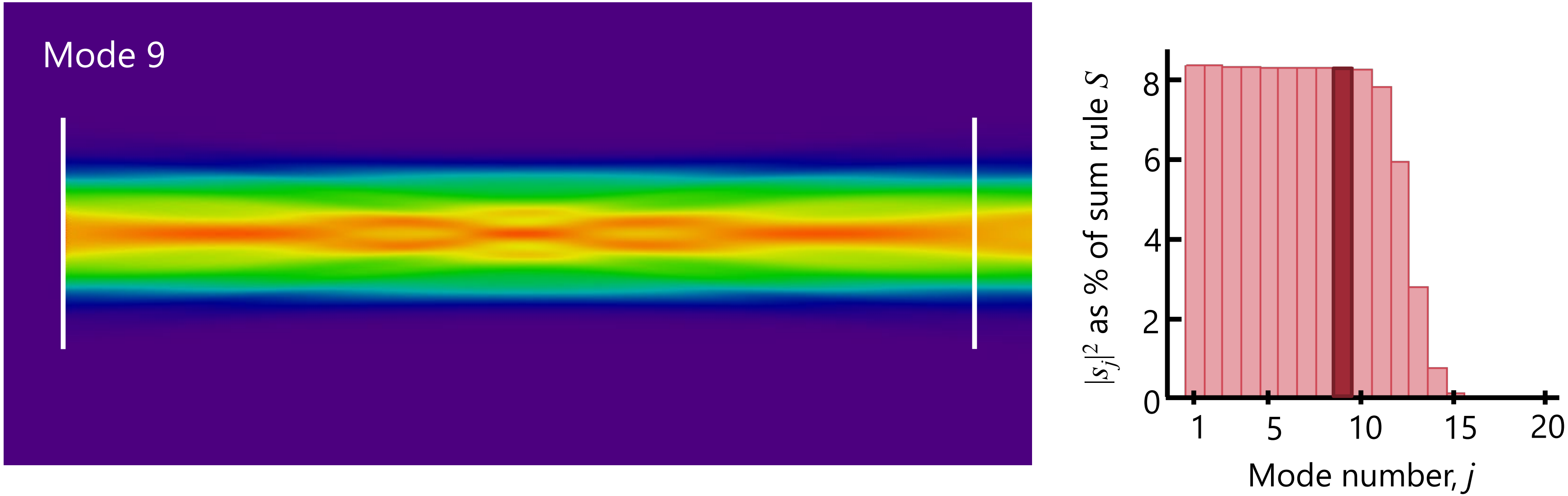
Mode 1
$|s_j|^2$ as % of sum rule S
0
2
4
6
8
1
5
10
15
20
Mode number, j



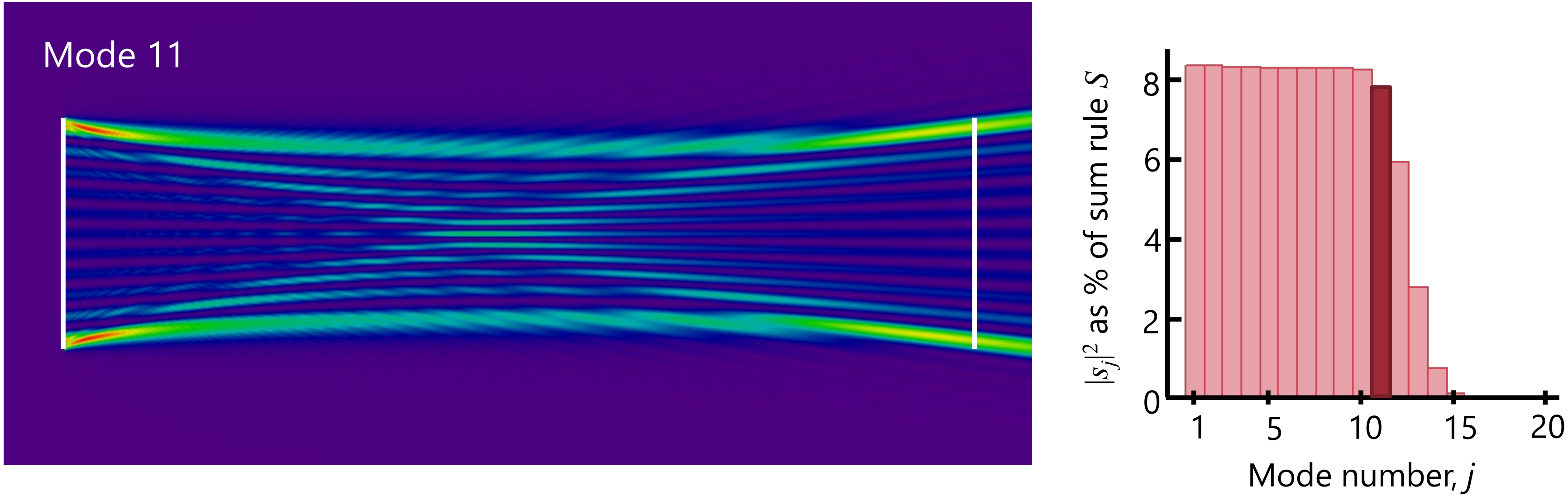
Mode 3
$|s_j|^2$ as % of sum rule S
0
2
4
6
8
1
5
10
15
20
Mode number, j


Mode 5
$|s_j|^2$ as % of sum rule S
0
2
4
6
8
1
5
10
15
20
Mode number, j

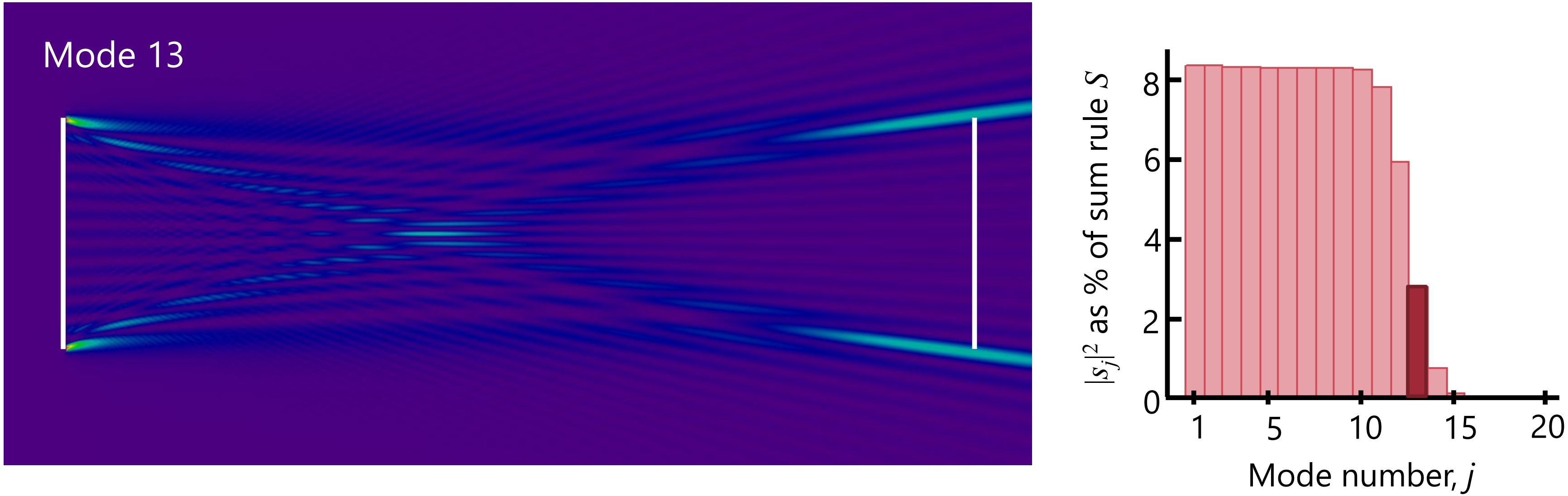

Mode 7
8
6
4
2
0
1
5
10
15
20
$|s_j|^2$ as % of sum rule S
Mode number, j



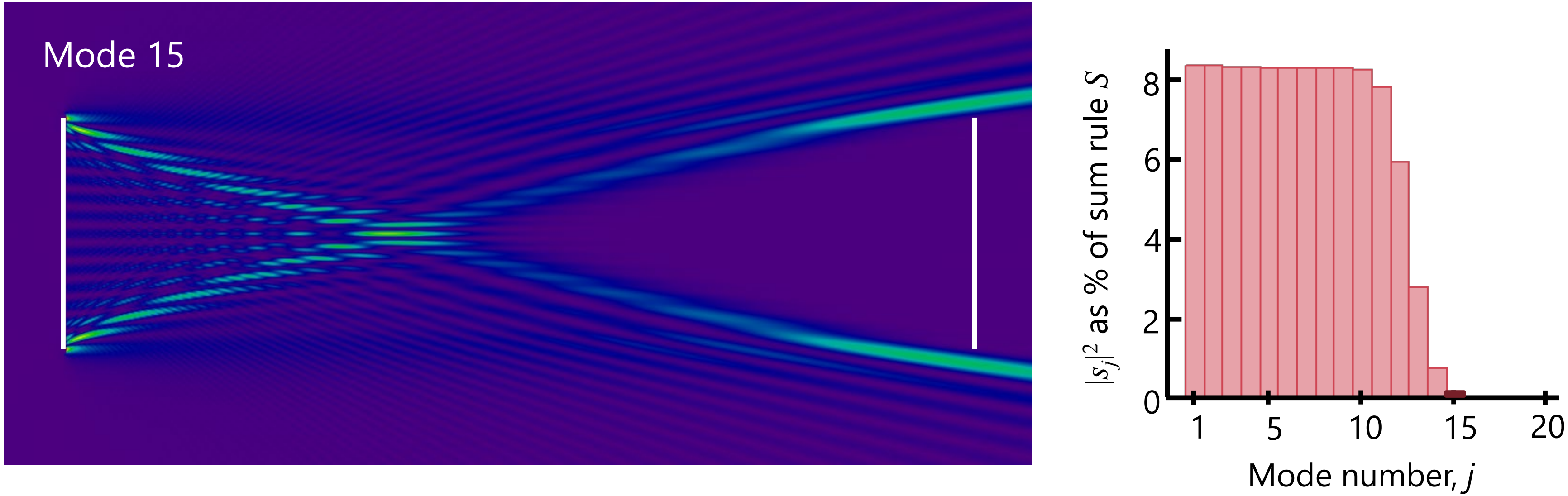
Mode 9
8
6
4
2
0
1
5
10
15
20
Mode number, j
|s_j|^2 as % of sum rule S



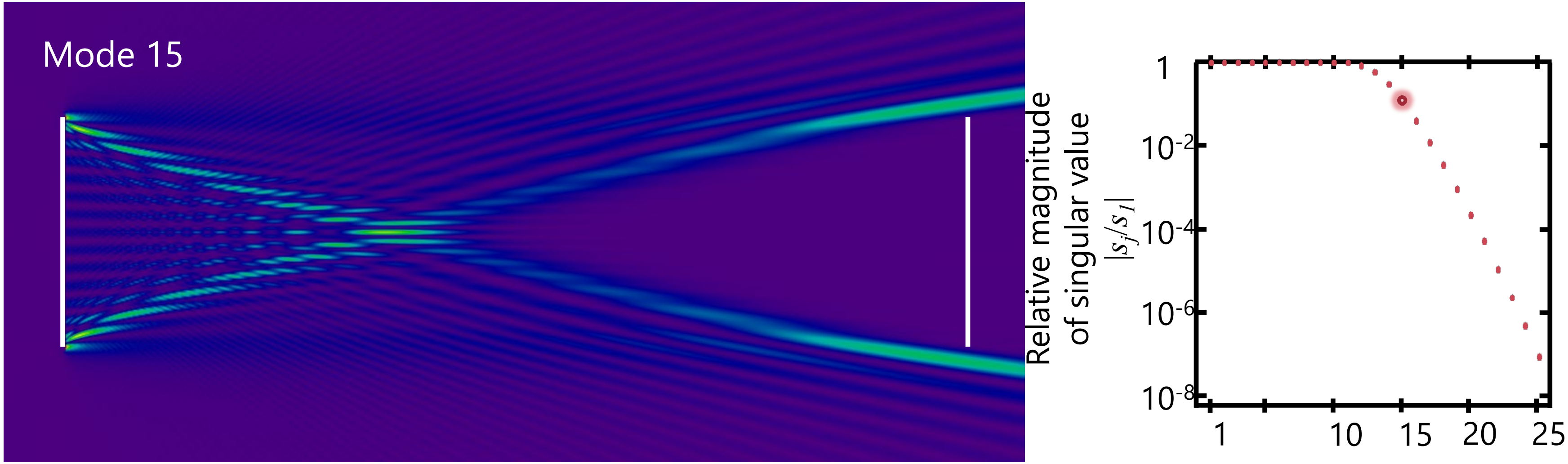
Mode 11
$|s_j|^2$ as % of sum rule S
0
2
4
6
8
1
5
10
15
20
Mode number, j



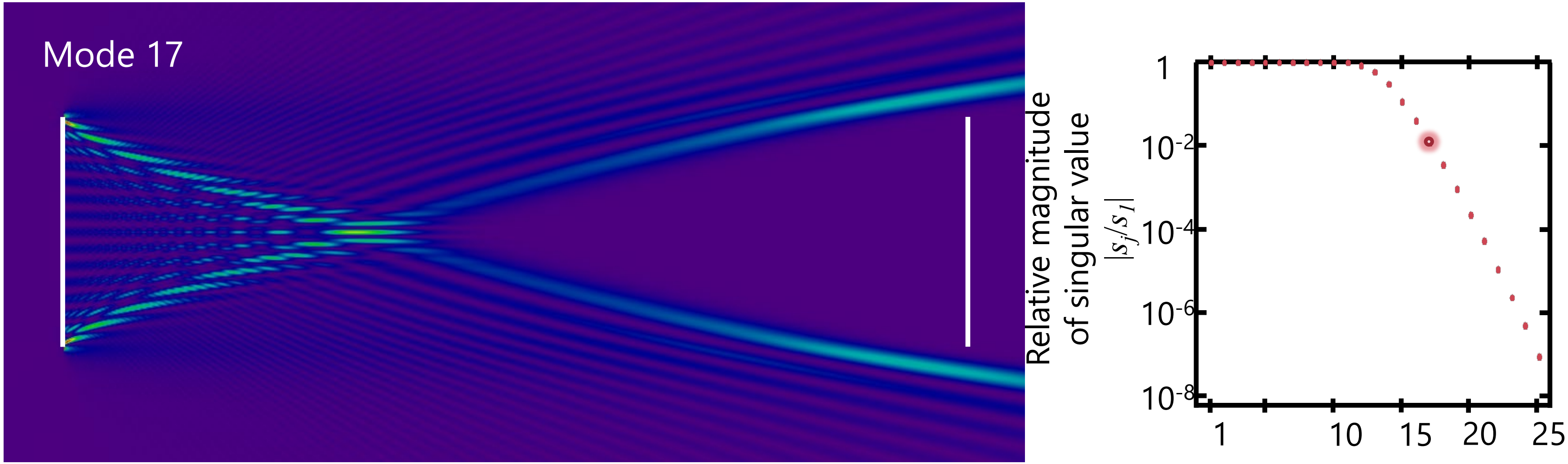
Mode 13
$|s_j|^2$ as % of sum rule S
0
2
4
6
8
1
5
10
15
20
Mode number, j



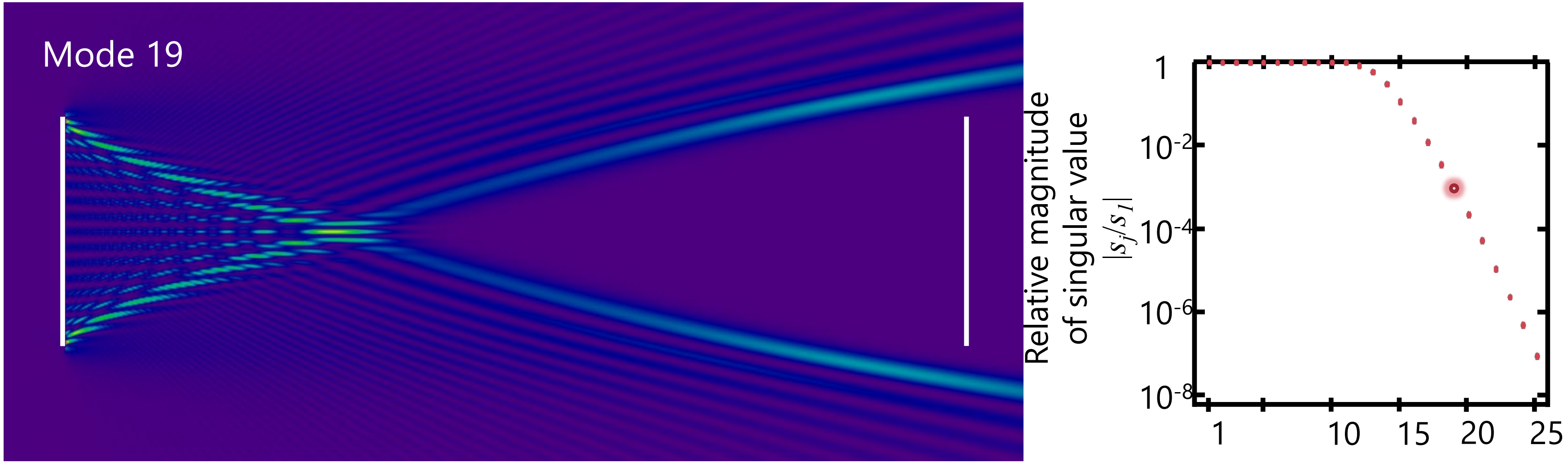
Mode 15
$|s_j|^2$ as % of sum rule S
0
2
4
6
8
1
5
10
15
20
Mode number, j



Mode 15
Relative magnitude of singular value
$|s_j/s_1|$
1
10^{-2}
10^{-4}
10^{-6}
10^{-8}
1
10
15
20
25

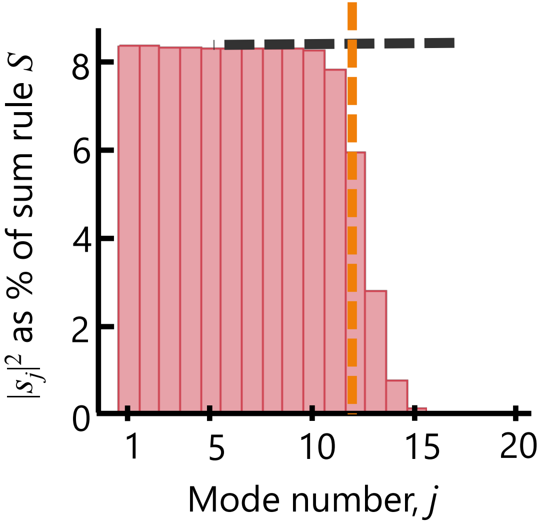


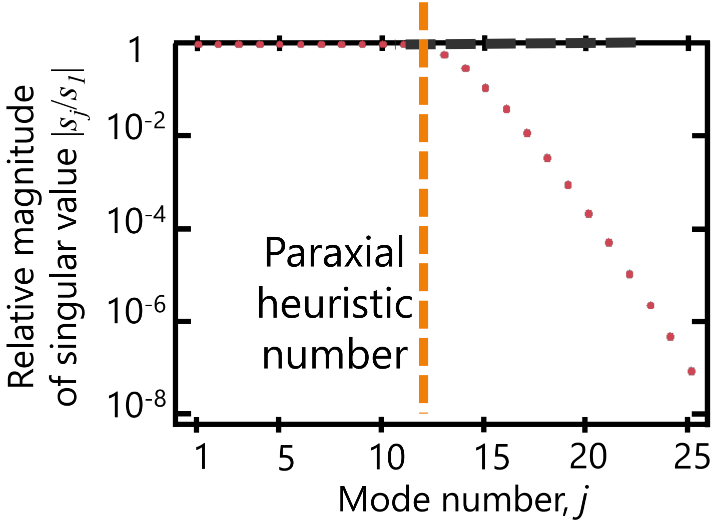
Mode 17
Relative magnitude
of singular value
$|s_j/s_1|$
1
10^{-2}
10^{-4}
10^{-6}
10^{-8}
1
10
15
20
25



Mode 19
Relative magnitude of singular value
$|s_j/s_1|$
1
10^{-2}
10^{-4}
10^{-6}
10^{-8}
1
10
15
20
25

Paraxial heuristic number and paraxial degeneracy





"Waves, modes, communications and optics," Adv. Opt. Photon. 11, 679 (2019)

Paraxial heuristic number

$N_H \sim W_S W_R / \lambda L$

for source and receiver widths W_S, W_R

separation L

wavelength λ

Once we pass the number we expect from conventional "diffraction limits"

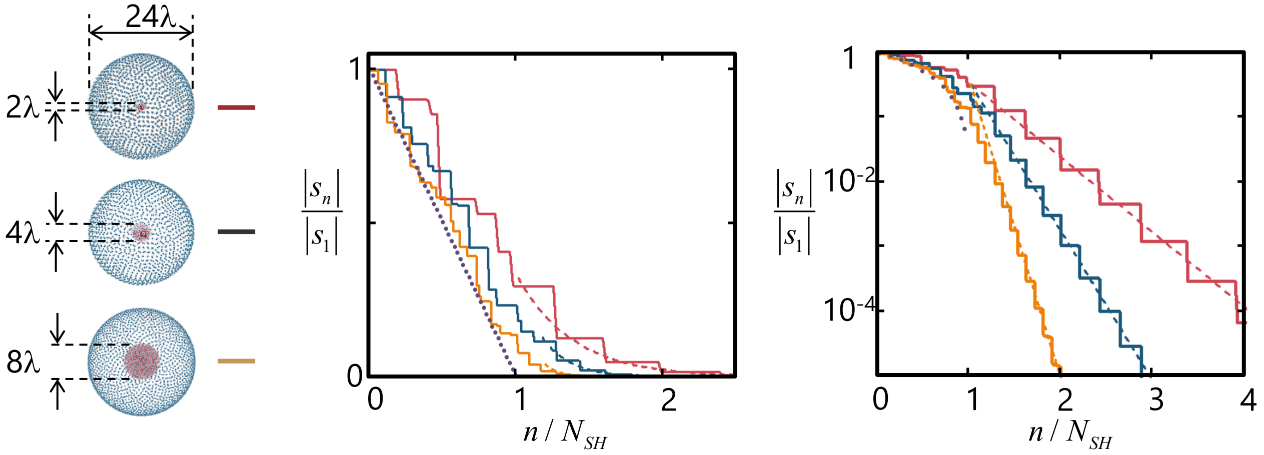
coupling strengths for further communication modes

fall off drastically and somewhat exponentially

We might think this is because the waves "miss" the receiving space

but that is not the general explanation

3D examples – concentric spherical shells



"Waves, modes, communications and optics," Adv. Opt. Photon. 11, 679 (2019)

Z. Kuang, D. A. B. Miller, and O. D. Miller, "Bounds on the Coupling Strengths of Communication Channels and Their Information Capacities," https://doi.org/10.48550/arXiv.2205.05150

Concentric spherical shell source and receiver spaces

are not easily analyzed by conventional "diffraction limit" theories

and do not show "paraxial degeneracy"

and the waves from the source space cannot "miss" the receiving space

but we still get some characteristic number of well-coupled communication modes

and a quasi-exponential fall-off of coupling beyond that

Why the abrupt fall-off past some number

- Why do we *always* see
  - regardless of the shape of the source and receiving volumes or surfaces
    - some number of “well coupled” channels
      - followed by an abrupt, quasi-exponential fall-off past this number
        - and just what gives this number?
- We might argue this is just “diffraction”
  - though that does not explain the concentric spheres case
    - where the waves cannot “miss” the receiving volume
- Is there some underlying piece of physics we are missing?

Tunneling escape of waves

stanford.io/3WSnn0S

David Miller, *Stanford University*
Zeyu Kuang, Owen Miller, *Yale University*

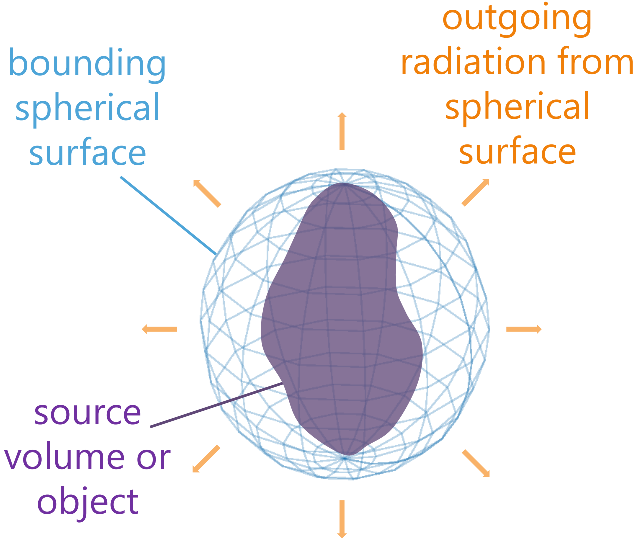
Waves from arbitrary volumes

How can we count the maximum number of well coupled waves (at a given frequency)

- from some finite volume?

Our approach

- Surround the volume with a mathematical "bounding" spherical surface
  - Count the number of well-coupled waves possible from this spherical surface
    - which then becomes the upper bound for waves from the source volume



D. A. B. Miller, Z. Kuang, O. D. Miller, "Tunneling escape of waves," http://arxiv.org/abs/2311.02744

Waves from arbitrary volumes

We show that, for spherical waves

with one key mathematical trick

there is a very simple and physical result

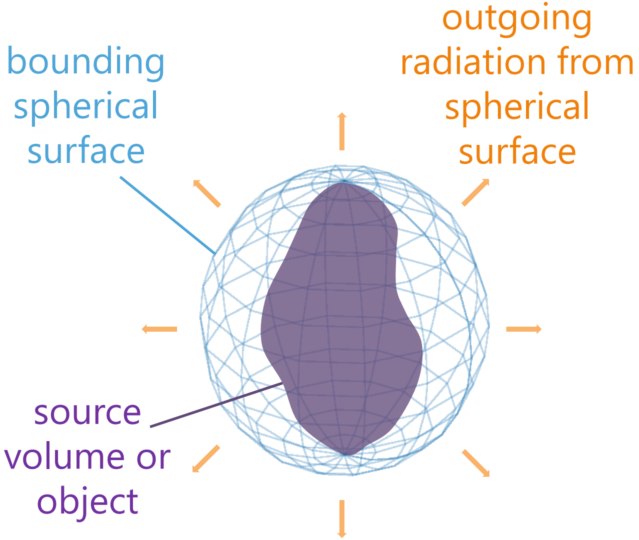
Beyond a certain simple threshold of "complexity" of spherical waves

they must "tunnel" to escape

Because the fall-off from tunneling is generally so rapid

this threshold effectively tells us the maximum number of well-coupled waves

and explains the quasi-exponential fall-off



D. A. B. Miller, Z. Kuang, O. D. Miller, "Tunneling escape of waves," http://arxiv.org/abs/2311.02744

Waves in spherical coordinates

In spherical coordinates r, θ, and ϕ

the solution to the wave equation separates to

$$U_{nm}(\mathbf{r}) = z_n(kr)Y_{nm}(\theta,\phi)$$

where $z_n(kr)$ is one of the spherical Bessel functions of order n, and

$Y_{nm}(\theta,\phi)$ is a spherical harmonic

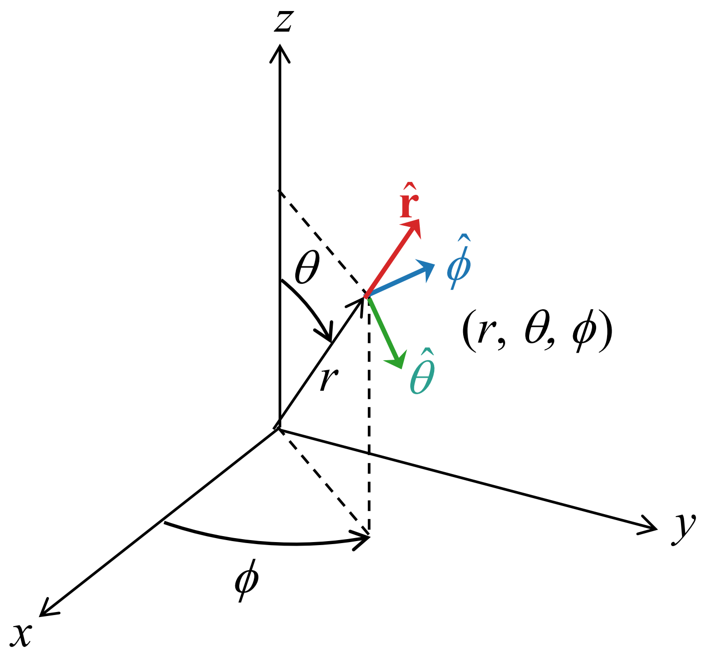
Here m and n are integers with

$n = 0,1,2,\ldots$ and $-n \le m \le n$

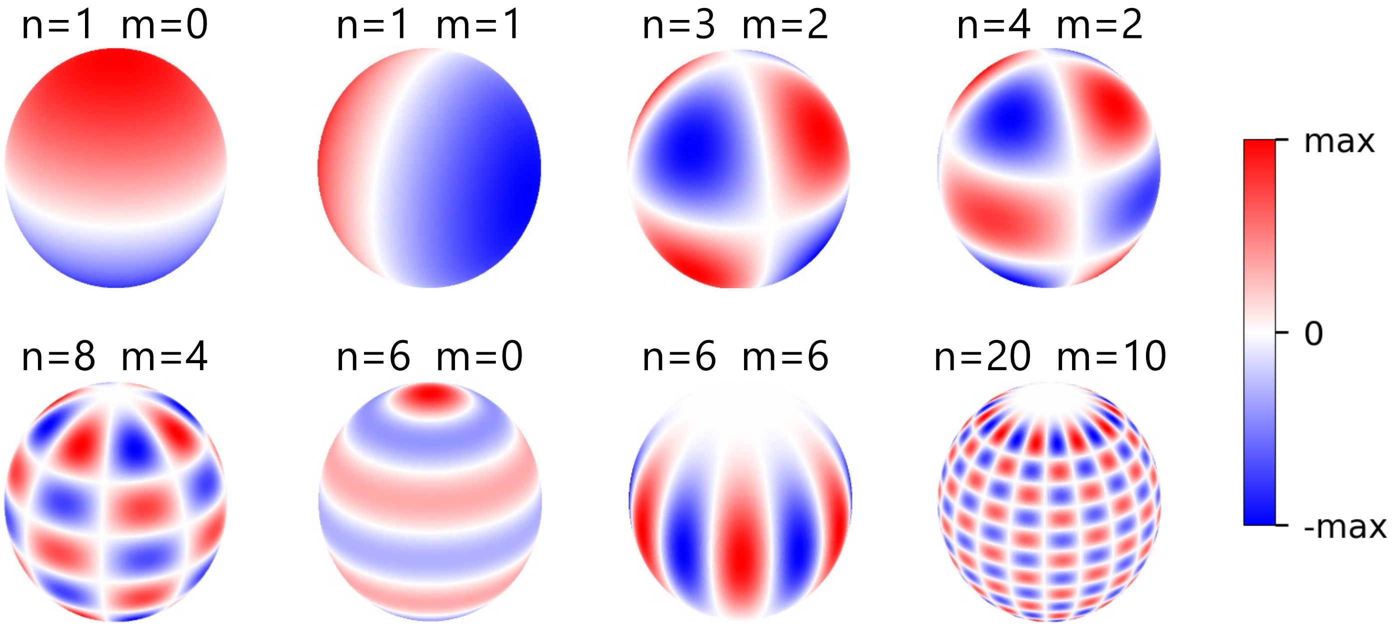
So, if we know the largest n for waves to propagate without tunneling

we can easily add up the total number of waves up to and including that n

$2n+1$ for each n



Spherical harmonics



Spherical harmonics are functions of angle only, and can be plotted on a spherical surface

They have *n* nodal circles altogether, with |*m*| through the poles (in their real form)

Escape radius

Specifically, for a given "order" n of spherical wave

there is an "escape radius"

$$r_{escn} = \frac{\sqrt{n(n+1)}}{k} \equiv \frac{\lambda_o}{2\pi}\sqrt{n(n+1)}$$

So, if the radius r_o of the spherical surface of interest

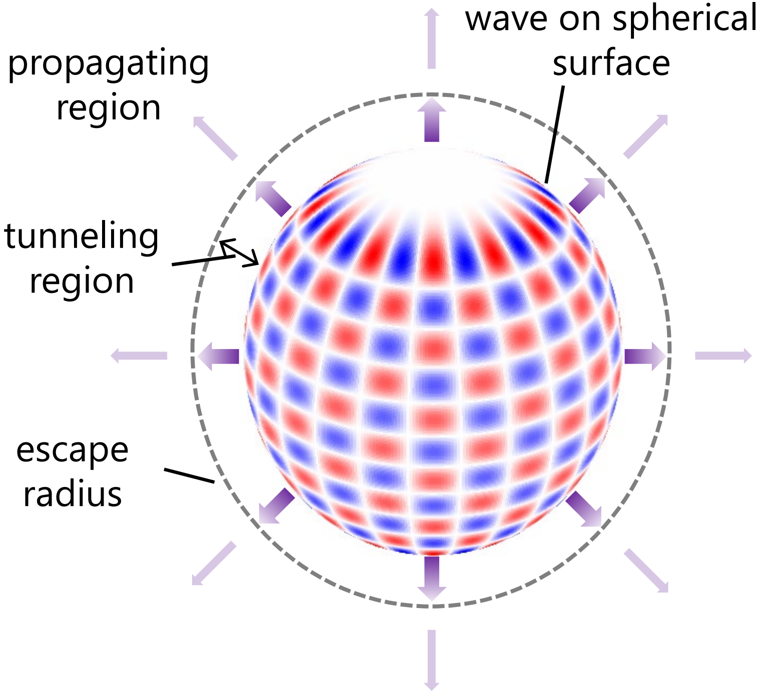
is smaller than the escape radius

for some order n of spherical wave

a wave with this n must tunnel

until it reaches the escape radius

after which it can propagate



D. A. B. Miller, Z. Kuang, O. D. Miller, "Tunneling escape of waves," http://arxiv.org/abs/2311.02744

Spherical Bessel functions and equation

Spherical Bessel functions obey

$$\rho^2 \frac{d^2 z_n(\rho)}{d\rho^2} + 2\rho \frac{dz_n(\rho)}{d\rho} + \left(\rho^2 - n(n+1)\right) z_n(\rho) = 0$$

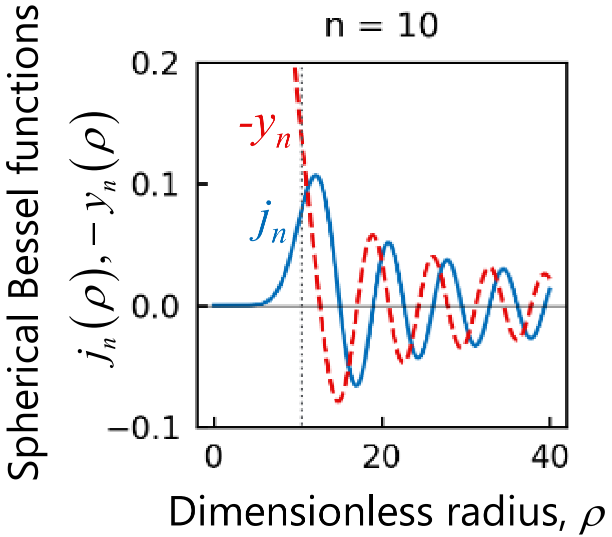
Classic radial standing wave solutions are

j_n which grows quasi-exponentially for small radii
and is quasi-oscillatory for larger radii

y_n which is singular at the origin
decaying quasi-exponentially for small radii
becoming quasi-oscillatory at large radii

Physically, ρ here is the dimensionless radius

$$\rho = kr \equiv 2\pi \frac{r}{\lambda}$$



Taking out the 1/radius dependence

Since the spherical Bessel functions have an underlying 1/radius dependence at large radius

- as appropriate for what are ultimately spherically expanding waves

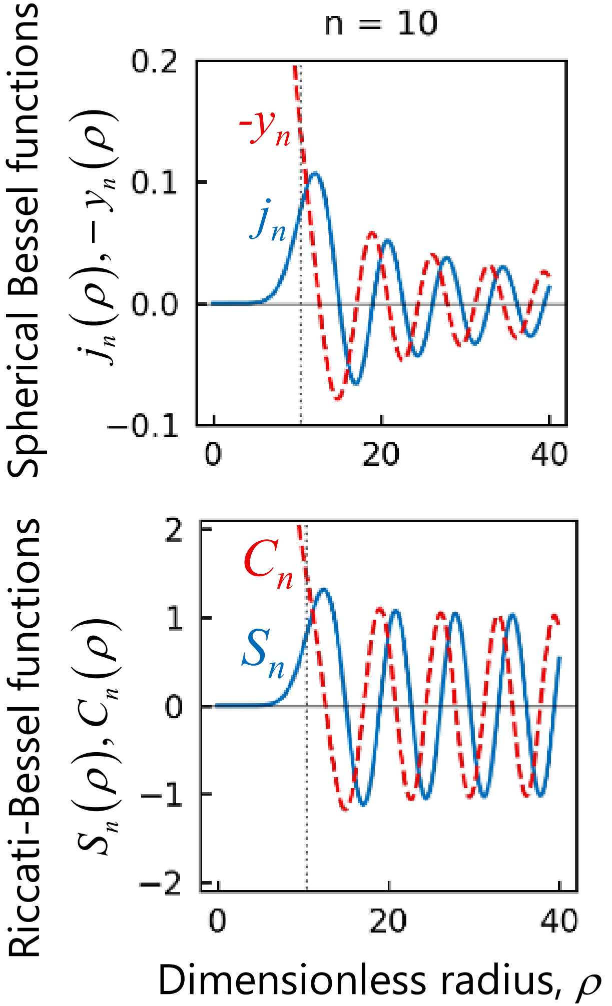
- it could be useful to remove that dependence
  - multiplying by radius
  - which gives functions corresponding to power per unit solid angle
  - rather than power per unit area

So, we recast in terms of such functions

- known as Riccati-Bessel functions

$$S_n(\rho) = \rho j_n(\rho) \qquad C_n(\rho) = -\rho y_n(\rho)$$

$$\xi_n(\rho) = \rho h_n^{(1)}(\rho) \equiv S_n(\rho) - iC_n(\rho)$$



Riccati-Bessel equation

Given that the spherical Bessel functions satisfy

$$\rho^2 \frac{d^2 z_n(\rho)}{d\rho^2} + 2\rho \frac{dz_n(\rho)}{d\rho} + \left(\rho^2 - n(n+1)\right) z_n(\rho) = 0$$

then we can easily check that all the Riccati-Bessel functions satisfy

$$\rho^2 \frac{d^2 \zeta_n}{d\rho^2} + \left(\rho^2 - n(n+1)\right)\zeta = 0$$

We can rearrange that to

$$-\frac{d^2 \zeta_n}{d\rho^2} + \frac{n(n+1)}{\rho^2}\zeta_n = \zeta_n$$

Riccati-Bessel "Schrödinger" equation

But wait!!!!!!

$$-\frac{d^2\zeta_n}{d\rho^2}+\frac{n(n+1)}{\rho^2}\zeta_n=\zeta_n$$

is in the form of a Schrödinger equation

$$-\frac{d^2\zeta_n}{d\rho^2}+V(\rho)\zeta_n=E_n\zeta_n$$

with effective radial potential

$$V(\rho)=\frac{n(n+1)}{\rho^2}$$

and the same "eigenenergy" $E_n=1$ for all n

Tunneling escape and escape radius

With the equation
$$-\frac{d^2\zeta_n}{d\rho^2}+\frac{n(n+1)}{\rho^2}\zeta_n=\zeta_n$$

if the "potential energy" exceeds the "total energy", i.e., if

$$\frac{n(n+1)}{\rho^2}>1 \quad \text{or, equivalently} \quad n(n+1)>\rho^2$$

the wave will be tunneling rather than propagating

So, for each n, there is an "escape radius"
$$\rho_{escn}=\sqrt{n(n+1)}$$

or, equivalently, in dimensioned form
$$r_{escn}=\frac{\sqrt{n(n+1)}}{k}\equiv\frac{\lambda_o}{2\pi}\sqrt{n(n+1)}$$

Evanescent and spherical escaping waves

Both plane and spherical waves start with the same tunneling barrier height

and hence the same initial decay

but the falling barrier height for the spherical wave

means it eventually escapes

to being a propagating wave

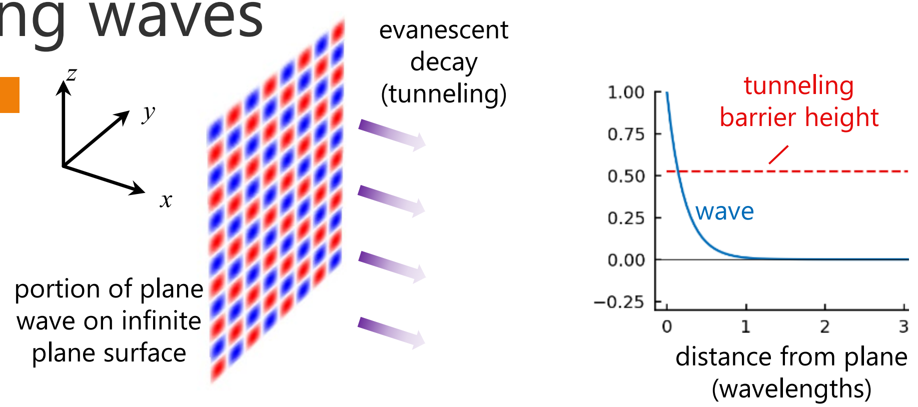
Note *all* such spherical waves

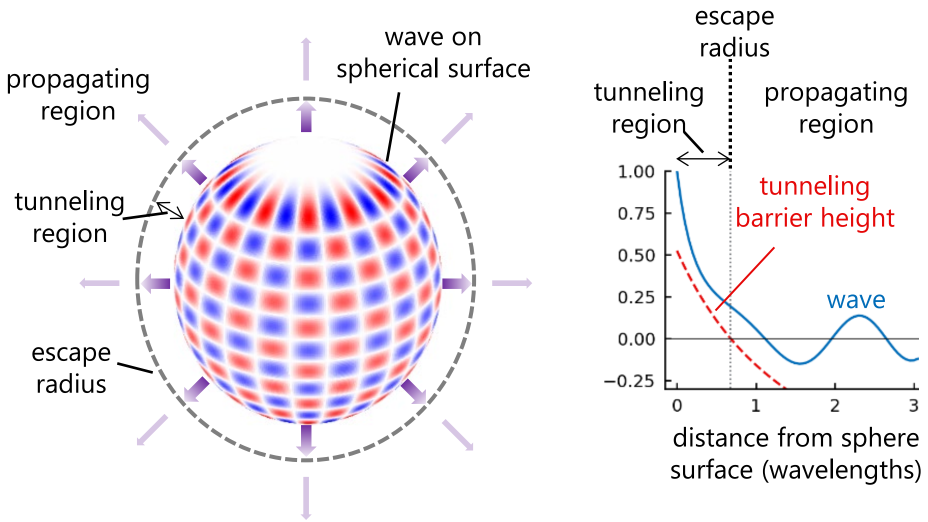
eventually escape to some degree

though the evanescent plane wave

never does

This is an artifact of the "infinite" extent of the plane wave





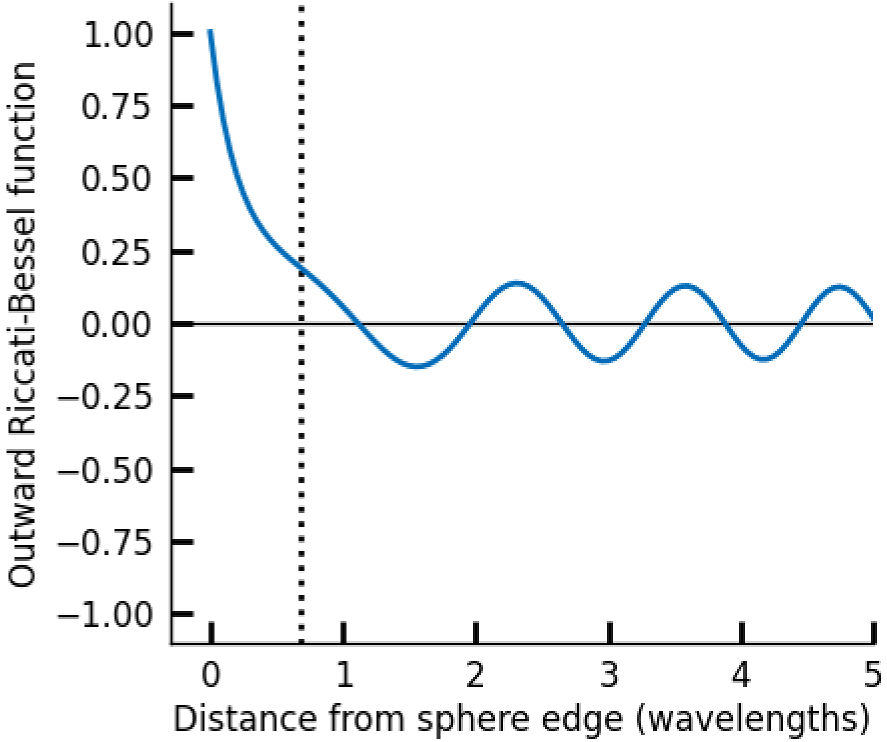
Outward wave propagation

As time progresses
- the wave beyond the escape radius
  - propagates outwards

We plot the outward Riccati-Bessel wave
- as a function of time
  - technically the real part of

$$\xi_n(2\pi r)\exp(-i\omega t)$$

  - normalized to unit amplitude at the sphere edge
- for a sphere of radius 2.9 wavelengths
  - with $n = 22$
    - which has an escape radius of 3.58 wavelengths



$r_o = 2.9 \quad n = 22 \quad r_{escn} = 3.58$

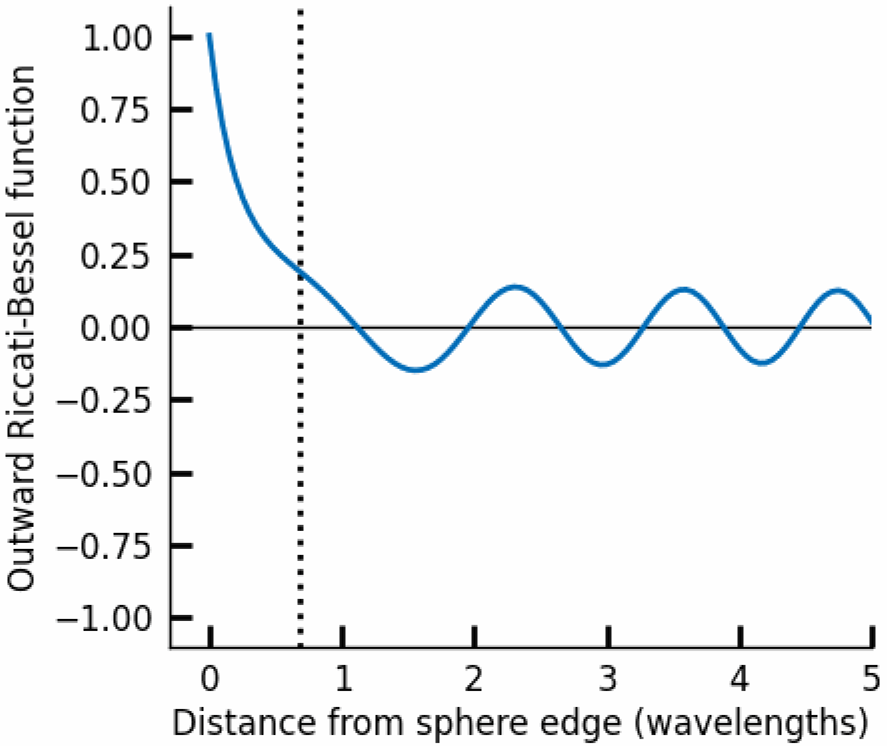
Outward wave propagation

As time progresses
- the wave beyond the escape radius
  - propagates outwards

We plot the outward Riccati-Bessel wave
- as a function of time
  - technically the real part of

$$\xi_n(2\pi r)\exp(-i\omega t)$$

  - normalized to unit amplitude at the sphere edge
- for a sphere of radius 2.9 wavelengths
  - with $n = 22$
    - which has an escape radius of 3.58 wavelengths



$r_o = 2.9 \quad n = 22 \quad r_{escn} = 3.58$

Spherical heuristic number

The threshold for tunneling is easy to characterize

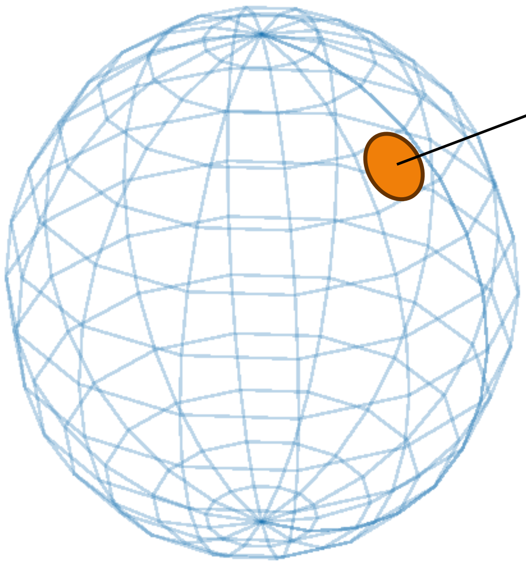
and gives a simple answer for the number of waves that do not need to tunnel

This is well approximated by the spherical heuristic number

$$N_{SH} = (kr_o)^2 \equiv \left(\frac{2\pi r_o}{\lambda_o}\right)^2 \equiv \frac{4\pi r_o^2}{(\lambda_o^2 / \pi)} \equiv \frac{A_S}{(\lambda_o^2 / \pi)}$$

where A_S is the sphere area

so one "propagating" wave for every λ_o^2 / π of surface area



D. A. B. Miller, Z. Kuang, O. D. Miller, "Tunneling escape of waves," http://arxiv.org/abs/2311.02744

Defining the diffraction limit

We can now construct a precise definition of
the "diffraction limit"

For a wave interacting with a volume
the wave passes the diffraction limit
if any spherical component of the wave must tunnel to enter or leave the bounding spherical surface enclosing the volume

Electromagnetic spherical outgoing waves

These have two transverse forms, separable in radial and angular parts

with the radial parts being the same as for the scalar case, so with

the same spherical/Riccati-Bessel tunneling and propagating behavior

and the angular part being a vector spherical harmonic function

$$\mathbf{C}_{mn}(\theta,\phi) = \nabla \times \left[\mathbf{r} Y_{nm}(\theta,\phi)\right] \equiv \nabla Y_{nm}(\theta,\phi) \times \mathbf{r} \qquad n = 1, 2\ldots \quad -n \le m \le n$$

giving "transverse electric" (TE) and "transverse magnetic" (TM) sets of waves

$$\mathbf{E}_{nm}^{(TE)}(r,\theta,\phi) = i\sqrt{\frac{\mu}{\varepsilon}}\, h_n^{(1)}(kr)\,\mathbf{C}_{mn}(\theta,\phi) \equiv i\sqrt{\frac{\mu}{\varepsilon}}\,\frac{\xi_n(kr)}{kr}\,\mathbf{C}_{mn}(\theta,\phi)$$

$$\mathbf{H}_{nm}^{(TM)}(r,\theta,\phi) = i\, h_n^{(1)}(kr)\,\mathbf{C}_{mn}(\theta,\phi) \equiv i\,\frac{\xi_n(kr)}{kr}\,\mathbf{C}_{mn}(\theta,\phi)$$

No n=0 electromagnetic outgoing waves

Note, because $\mathbf{C}_{mn}$ is a derivative of a spherical harmonic

and the spherical harmonic for $n = 0$ is uniform

there is no $n = 0$ wave in electromagnetism

If the first outgoing electromagnetic waves

(so, for $n = 1$)

are not to require tunneling to escape

the bounding spherical volume must be at least

$$r_{esc1} = \lambda_o / \left(\sqrt{2}\,\pi\right) \simeq 0.225\lambda_o$$

in radius or, equivalently, in diameter

$$d = \sqrt{2}\lambda_o / \pi \simeq 0.45\lambda_o$$

(consistent with the well-known Chu limit on antenna Q)

(Note: The escape radius for $n = 0$ acoustic waves is, however, zero

so, there is always one acoustic wave that can escape without tunneling

no matter how small the emitter or microphone)

Perfect cloaking - An optical “white hole”?

In this “white hole”, incoming light

- appears to be mostly “sucked” into the “white hole” in the middle
  - The phase fronts all “fall” rapidly into the “white hole”
- and then the light is regenerated
  - The phase fronts rapidly re-emerge from the “white hole”

How do we make this optical “white hole”?

- Note: it may be simpler than you think

Perfect cloacking - An optical "white hole"?

So, what does it take to build this cloak?

Absolutely nothing

- at least for this wave

If the wave is too complicated

- i.e., if it is trying to violate the "diffraction limit"
  - it can't even effectively get into the volume
    - and it "reflects off free space"

This is the "inward wave" version of the tunneling escape

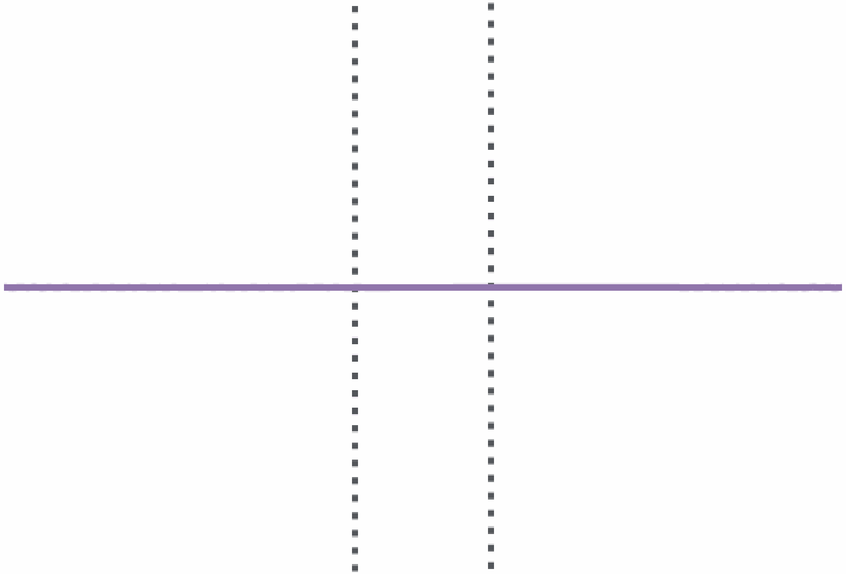
- with the wave trying to tunnel to get in

Interestingly

- the pulse actually looks as if it propagates right through!

Perfect cloaking?

Watch the blue dot, which propagates at the usual "phase velocity" of the wave



It appears to move right through the volume at a constant speed

Conclusions

For a copy of these slides e-mail
dabm@stanford.edu

There is a unified way of thinking about waves

based on waves from a spherical surface

from the propagating and evanescent fields of large optics

to the multipole expansions of antennas and nanophotonics

This approach gives a clear intuition

based on the onset of spherical wave tunneling

that

- explains how many waves can easily get in or out of a volume

  and why the fall-off is so abrupt past this number
- gives a rigorous and precise diffraction limit definition
- can also derive previous heuristic results

D. A. B. Miller, Z. Kuang, O. D. Miller,
"Tunneling escape of waves,"
http://arxiv.org/abs/2311.02744

stanford.io/3WSnn0S

Funding from AFOSR
grant FA9550-21-1-0312