Thermal distributions 2

Systems in thermal contact

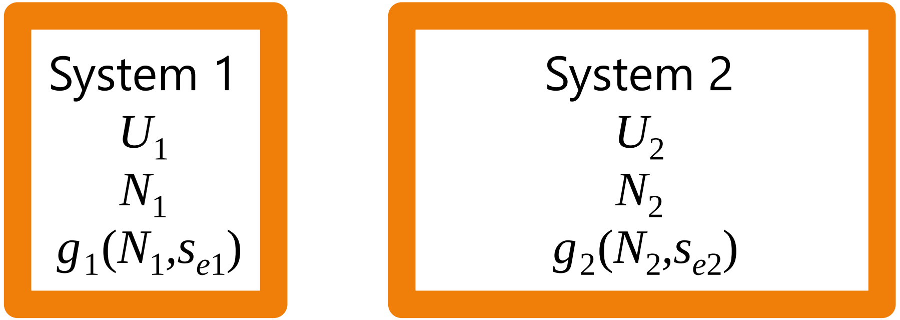
Modern physics for engineers

David Miller

Systems in thermal contact

Systems in thermal contact

- Now we presume two systems
  - each containing spins
    - each well insulated from the environment
- System 1 has energy U_1
  - N_1 spins, spin excess s_{e1}
    - and multiplicity $g\left(N_1, s_{e1}\right)$
- System 2 has energy U_2
  - N_2 spins, spin excess s_{e2}
    - and multiplicity $g\left(N_2, s_{e2}\right)$



System 1
U_1
N_1
$g_1(N_1, s_{e1})$

System 2
U_2
N_2
$g_2(N_2, s_{e2})$

Systems in thermal contact

We presume total energy in the combined system is conserved

- even when joined through a thermally conducting wall

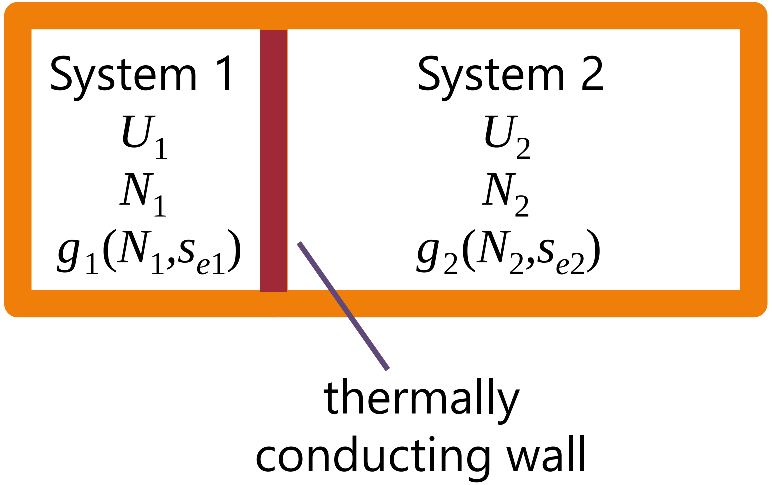
- So, at all times

$$s_e = s_{e1} + s_{e2}$$

- which gives the total energy through $U(s_e) = -2\mu_s B s_e$

Note N_1 and N_2 remain fixed

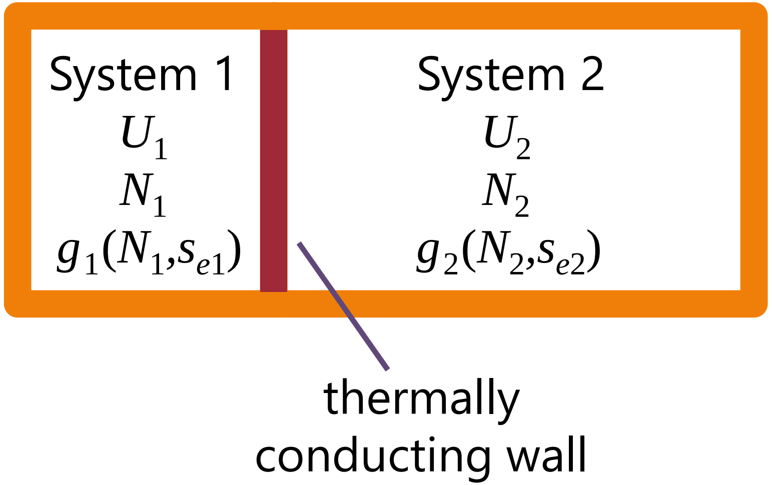
- Spins do not pass through the wall



Systems in thermal contact

Our goal now is to deduce

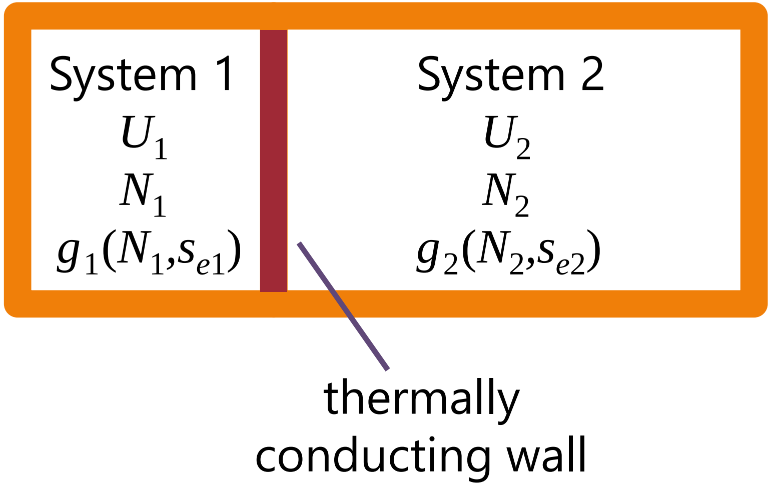
- what is the most likely macrostate
  - for splitting the total s_e
    - between the two systems
      - when we have reached thermal equilibrium
        - after joining the two systems through the thermally conducting wall



Systems in thermal contact

We know that, for large systems
- the most likely macrostate
  - and those close to it
    - dominate the possible microstates

So if we know this most likely macrostate and its properties
- we will essentially know the properties of the system at thermal equilibrium



Multiplicities for systems in thermal contact

Systems in thermal contact

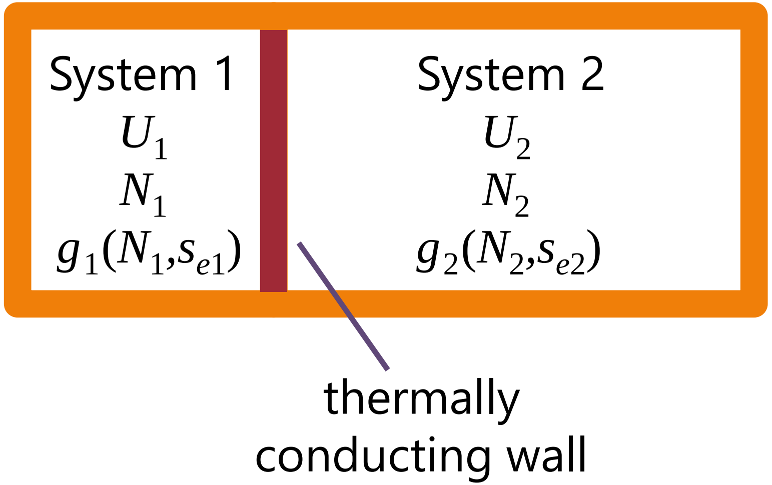
For a given s_{e1} and (large) N_1

we know the multiplicity of the macrostate for system 1

$$g\left(N_1, s_{e1}\right) \simeq g\left(N_1, 0\right) \exp\left(-\frac{2s_{e1}^2}{N_1}\right)$$

and similarly for system 2

$$g\left(N_2, s_{e2}\right) \simeq g\left(N_2, 0\right) \exp\left(-\frac{2s_{e2}^2}{N_2}\right)$$


System 1

U_1

N_1

$g_1(N_1, s_{e1})$

System 2

U_2

N_2

$g_2(N_2, s_{e2})$

thermally conducting wall

Systems in thermal contact

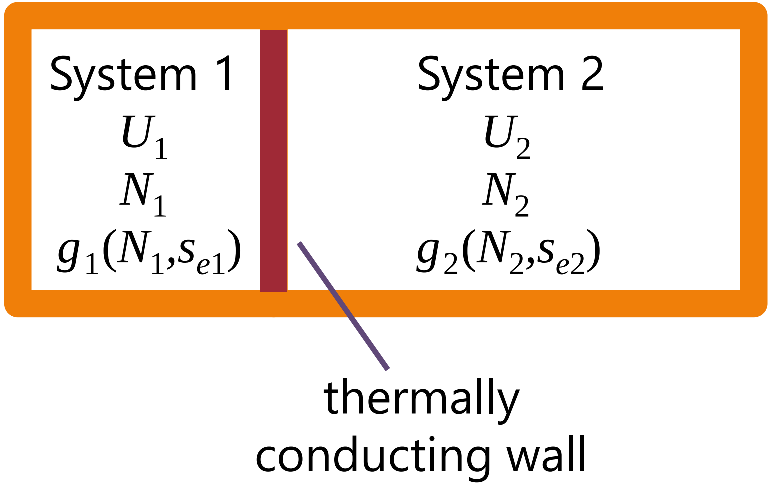
The multiplicity

of the macrostate

of the entire system

is the product of

the multiplicities of the parts



System 1

U_1

N_1

$g_1(N_1, s_{e1})$

System 2

U_2

N_2

$g_2(N_2, s_{e2})$

thermally conducting wall

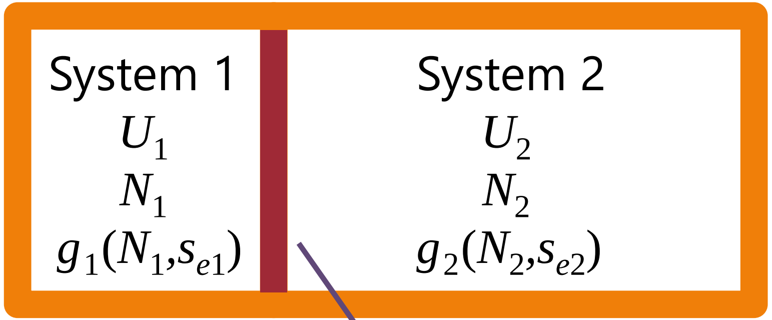
Systems in thermal contact

That is, in given macrostates of systems 1 and 2

- if for each microstate of system 1
  - there are g_2 possible microstates of system 2
- and we know there are g_1 possible microstates of system 1

then the total number of possible microstates for the entire system is

$$g_1 g_2$$



thermally conducting wall

Systems in thermal contact

With $g_1\left(N_1, s_{e1}\right) \simeq g\left(N_1, 0\right)\exp\left(-\frac{2s_{e1}^2}{N_1}\right)$

possible microstates for system 1

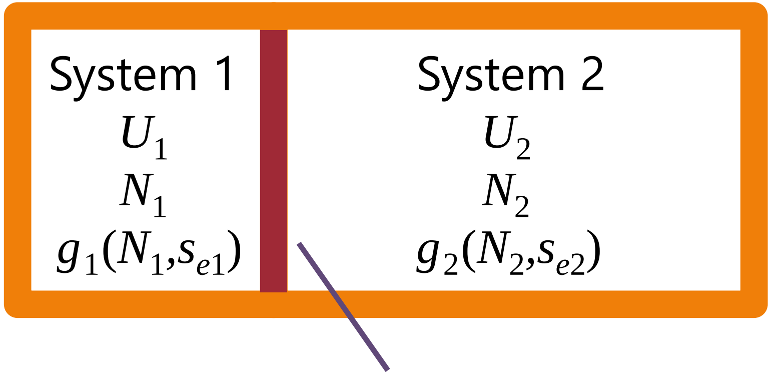
and $g_2\left(N_2, s_{e2}\right) \simeq g\left(N_2, 0\right)\exp\left(-\frac{2s_{e2}^2}{N_2}\right)$

possible microstates for system 2

then for the entire system, there are

$$g_{tot} \simeq g\left(N_1, 0\right) g\left(N_2, 0\right)\exp\left(-\frac{2s_{e1}^2}{N_1}\right)\exp\left(-\frac{2s_{e2}^2}{N_2}\right)$$

microstates in the macrostate



System 1
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thermally conducting wall

Thermal distributions 2

Maximizing multiplicities for spin systems

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Maximizing multiplicity

With this multiplicity

$$g_{tot} \simeq g(N_1,0)\,g(N_2,0)\exp\left(-\frac{2s_{e1}^2}{N_1}\right)\exp\left(-\frac{2s_{e2}^2}{N_2}\right)$$

for the combined system in thermal contact

now we want to understand how to split up

$$s_e = s_{e1} + s_{e2}$$

between the two systems

to maximize the total multiplicity g_{tot}

Maximizing multiplicity

Since s_e is fixed by conservation of energy overall

we can rewrite

$$g_{tot} \simeq g(N_1,0)\,g(N_2,0)\exp\left(-\frac{2s_{e1}^2}{N_1}\right)\exp\left(-\frac{2s_{e2}^2}{N_2}\right)$$

in terms of a fixed s_e and a variable s_{e1}

$$g_{tot}(s_{e1}) \simeq g(N_1,0)\,g(N_2,0)\exp\left(-\frac{2s_{e1}^2}{N_1}\right)\exp\left(-\frac{2(s_e - s_{e1})^2}{N_2}\right)$$

We could differentiate this with respect to s_{e1}

to find the choice that gives the maximum g_{tot}

Maximizing multiplicity

It is slightly easier to take the logarithm of

$$g_{tot}\left(s_{e1}\right) \simeq g\left(N_1,0\right)g\left(N_2,0\right)\exp\left(-\frac{2s_{e1}^2}{N_1}\right)\exp\left(-\frac{2\left(s_e-s_{e1}\right)^2}{N_2}\right)$$

that is

$$\log\left[g_{tot}\left(s_{e1}\right)\right] \simeq \log\left[g\left(N_1,0\right)g\left(N_2,0\right)\right]-\frac{2s_{e1}^2}{N_1}-\frac{2\left(s_e-s_{e1}\right)^2}{N_2}$$

and differentiate that

If a positive function is maximized

so also is its logarithm

and *vice versa*

Maximizing multiplicity

Differentiating

$$\log\left[g_{tot}\left(s_{e1}\right)\right] \simeq \log\left[g\left(N_1,0\right)g\left(N_2,0\right)\right] - \frac{2s_{e1}^2}{N_1} - \frac{2\left(s_e - s_{e1}\right)^2}{N_2}$$

and setting the result to zero gives

$$\frac{d\log\left[g_{tot}\left(s_{e1}\right)\right]}{ds_{e1}} \simeq -\frac{4s_{e1}}{N_1} + \frac{4\left(s_e - s_{e1}\right)}{N_2} = 4\left(\frac{s_{e2}}{N_2} - \frac{s_{e1}}{N_1}\right) = 0$$

so to maximize the multiplicity of the whole system

we choose $\dfrac{s_{e2}}{N_2} = \dfrac{s_{e1}}{N_1}$

Energy per spin

Since in a given magnetic field

the energy of a given one of the systems is just

proportional to s_{e1} or s_{e2}

that is, $U_1(s_{e1}) = -2\mu_s B s_{e1}$ and $U_2(s_{e2}) = -2\mu_s B s_{e2}$

the statement $\frac{s_{e2}}{N_2} = \frac{s_{e1}}{N_1}$ is equivalent to saying

the energy per spin in the two systems is the same

that is, $\frac{U_1}{N_1} = \frac{U_2}{N_2}$

Energy per spin

If the energy per spin is the same on each side

then the energy per spin in the entire system

is also the same

That is, with $U = U_1 + U_2$ and $N = N_1 + N_2$

$$\frac{U_1}{N_1} = \frac{U_2}{N_2} = \frac{U}{N}$$

and similarly

$$\frac{S_{e2}}{N_2} = \frac{S_{e1}}{N_1} = \frac{S_e}{N}$$

Systems in thermal equilibrium

So, by example

- we have uncovered a relatively straightforward result
  - The average energy for each similar microscopic system
    - here the spins
      - is the same for systems at thermal equilibrium

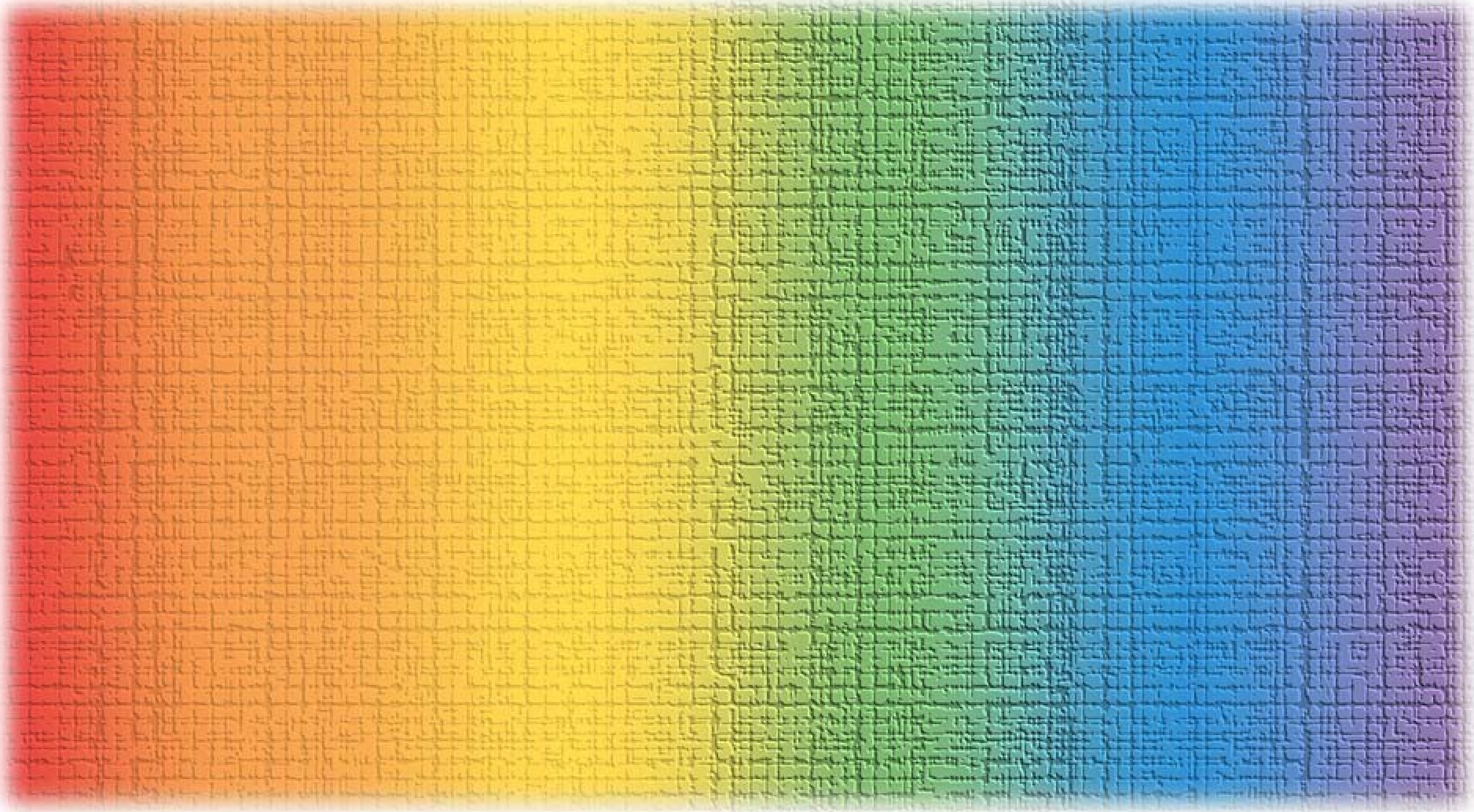
Systems in thermal equilibrium

To understand the deeper meaning of this

we need to generalize more

and introduce two more concepts

entropy and temperature



Thermal distributions 2

Maximizing multiplicity for general systems

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Two generalized systems

Now we generalize beyond just spin systems

to two systems with

N_1 and N_2 quantum “particles” of the same kinds on each side respectively

with $N = N_1 + N_2$

and with energies U_1 and U_2

with $U = U_1 + U_2$

We consider these systems to have multiplicities

$g_1(N_1, U_1)$ and $g_2(N_2, U_2)$

Multiplicity of the combined system

To obtain the total multiplicity $g(N,U)$

for the combined system

we should sum up the multiplicities

for every possible combination of U_1 and U_2

that corresponds to the same total energy U

$$g(N,U)=\sum_{U_1} g_1(N_1,U_1)\,g_2(N_2,U-U_1)$$

substituting $U-U_1$ for U_2

and summing over all possible values of U_1

Multiplicity of the combined system

We presume the multiplicity for the combined system

$$g(N,U)=\sum_{U_1} g_1(N_1,U_1)\,g_2(N_2,U-U_1)$$

is dominated by the term in the sum for which

the product $g_1(N_1,U_1)\,g_2(N_2,U-U_1)$ is largest

or at least a group of terms close to that

so we will consider just such terms

finding the conditions under which they are maximized

We presume this approach effectively maximizes $g(N,U)$

Maximizing the multiplicity

For such a product $g = g_1 g_2$

treating U_1 and U_2 as independent variables for the moment

$$\frac{\partial(g_1 g_2)}{\partial U_1} = \left(\frac{\partial g_1}{\partial U_1}\right) g_2 \text{ and } \frac{\partial(g_1 g_2)}{\partial U_2} = g_1 \left(\frac{\partial g_2}{\partial U_2}\right)$$

so the differential can be written

$$dg = \left(\frac{\partial g_1}{\partial U_1}\right)\Bigg|_{N_1} g_2 dU_1 + g_1 \left(\frac{\partial g_2}{\partial U_2}\right)\Bigg|_{N_2} dU_2$$

Maximizing the multiplicity

The idea of this differential

$$dg = \left.\left(\frac{\partial g_1}{\partial U_1}\right)\right|_{N_1} g_2 dU_1 + g_1 \left.\left(\frac{\partial g_2}{\partial U_2}\right)\right|_{N_2} dU_2$$

is that it would formally tell us

the small change in the multiplicity g

that would result from small changes in U_1 and U_2

presuming we knew the partial derivatives

Maximizing the multiplicity

Here we presume total energy is conserved

- so if U_1 increases by some amount
  - then U_2 must decrease by an equal amount
    - so $dU_1 = -dU_2$

If the system is to be in the macrostate with the largest multiplicity

- then its multiplicity $g = g_1 g_2$ should be at a maximum as far as the choice of U_1 is concerned
  - Therefore, an infinitesimally small change in U_1
    - should make no change in $g_1 g_2$

Maximizing the multiplicity

Hence the differential of $g = g_1 g_2$ should be zero

and using $dU_1 = -dU_2$

we have

$$dg = \left(\frac{\partial g_1}{\partial U_1}\right)\Bigg|_{N_1} g_2 dU_1 - g_1 \left(\frac{\partial g_2}{\partial U_2}\right)\Bigg|_{N_2} dU_1 = 0$$

from which we conclude

$$\frac{1}{g_1}\left(\frac{\partial g_1}{\partial U_1}\right)\Bigg|_{N_1} = \frac{1}{g_2}\left(\frac{\partial g_2}{\partial U_2}\right)\Bigg|_{N_2}$$

Maximizing the multiplicity

Equivalently to $\frac{1}{g_1}\left(\frac{\partial g_1}{\partial U_1}\right)\Bigg|_{N_1} = \frac{1}{g_2}\left(\frac{\partial g_2}{\partial U_2}\right)\Bigg|_{N_2}$

we can write $\left(\frac{\partial \log g_1}{\partial U_1}\right)\Bigg|_{N_1} = \left(\frac{\partial \log g_2}{\partial U_2}\right)\Bigg|_{N_2}$

So, we are concluding that

in thermal equilibrium

these two partial derivatives

should be equal

So perhaps $\log g$ has some meaning?