Oscillations and waves 2

Eigen equations and operators

Modern physics for engineers

David Miller

Eigen equations

Eigen equations

Note that the equation for the simple harmonic oscillator

$$\frac{d^2y}{dt^2} = -\frac{K}{M}y = -\omega^2 y$$

is of the form

Operator operating on the function $y(t)$ = a constant times the function $y(t)$

where here the operator is d^2/dt^2

and the constant is $-\omega^2 = -K/M$

Note that an "operator" is

something that turns one function into another function

Eigen equations

If we denote an operator by A

and functions by $\mathbf{y}$ and $\mathbf{z}$ respectively

then we can introduce a mathematical notation $\mathrm{A}\mathbf{y} = \mathbf{z}$

where we mean that operator A operating on function $\mathbf{y}$ gives rise to function $\mathbf{z}$

So instead of

operator operating on the function $y(t)$ = a constant times the function $y(t)$

we can write the equation $\mathrm{A}\mathbf{y} = b\mathbf{y}$

for some constant b

Eigen equations

The form of equation given by

Operator operating on the function $y(t)$ = a constant times the function $y(t)$

or $\mathrm{A}\mathbf{y} = b\mathbf{y}$

is called an eigen equation

Any such constant b for which such an equation holds is

an eigenvalue

and a function that is a solution for that eigenvalue is

an eigenfunction

associated with that eigenvalue

Eigen equations

The particular eigen equation $\frac{d^2y}{dt^2} = -\frac{K}{M}y$

- for our physical problem of a mass on a spring
  - for a given K and M

physically is only allowed to have one eigenvalue ($-K/M$)

- and one eigenfunction
  - at least if we allow different amplitudes and phases

This particular oscillator

- the simple harmonic oscillator
  - is an oscillator that physically only has one mode

Linear operators

Linear operators

An important point is that the operator d^2 / dt^2

is what is called a linear operator

A linear operator should obey the following relations

(i) Operator operating on (b times y) = b times (Operator operating on y)

for an arbitrary constant b and a function y

(ii) Operator operating on ($y_1 + y_2$) = (Operator operating on y_1) + (Operator operating on y_2)

for any two functions y_1 and y_2

Linear operators

In our more abstract notation, these conditions become

$$\mathrm{A}b\mathbf{x} = b\mathrm{A}\mathbf{x}$$

and

$$\mathrm{A}(\mathbf{x}_1 + \mathbf{x}_2) = \mathrm{A}\mathbf{x}_1 + \mathrm{A}\mathbf{x}_2$$

Our abstract notation for such linear operators

is exactly the same as matrix and vector notation

We can think of A as a matrix, and entities like $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ as being column vectors

Quite generally

linear operators can be represented by matrices

Oscillations and waves 2

The classical wave equation

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The classical wave equation

Waves are very common in classical physics

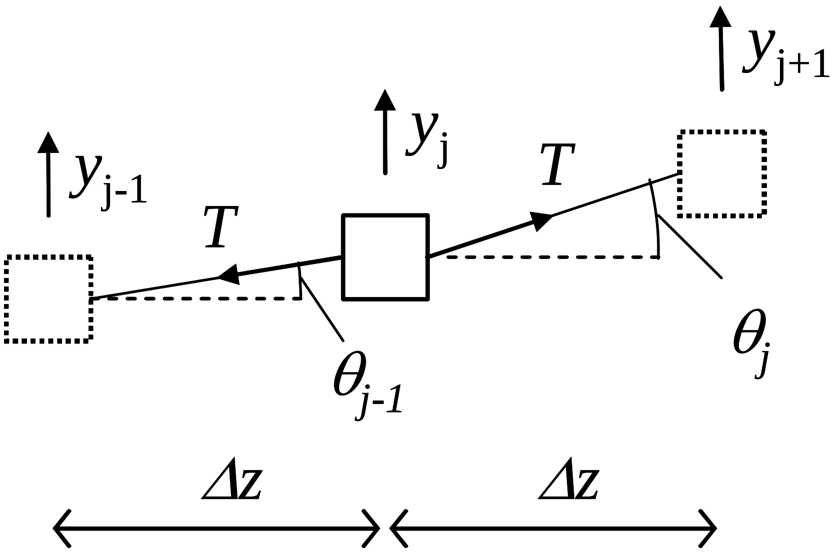
- sound waves through air
- waves on the surface of water
- electromagnetic waves that we exploit for radio and light

and we will also encounter quantum mechanical waves

Here, by looking at waves on a string

we introduce a simple example of a classical wave equation

Classical wave equation

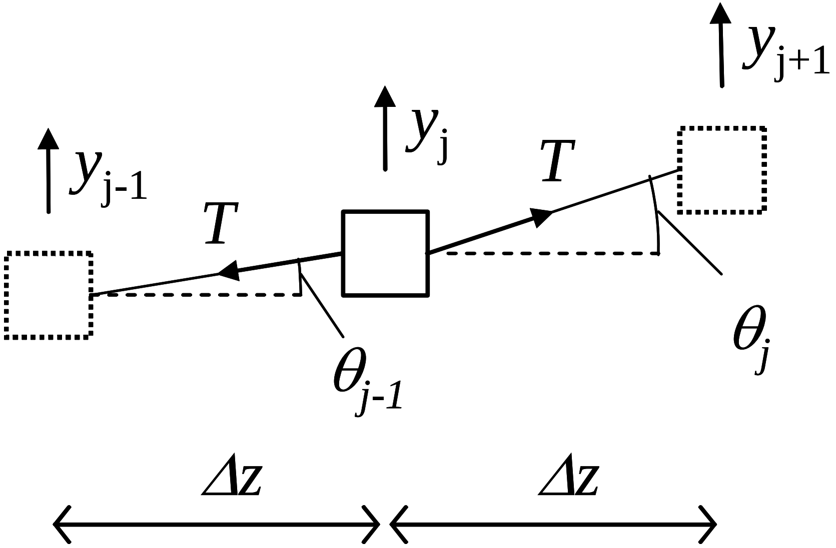


Imagine a large number of identical masses connected by a string under a tension T

- The string itself is presumed to have effectively no mass

Here we look at the jth mass and the forces on it

- The masses have vertical displacements y_j at some time



Classical wave equation

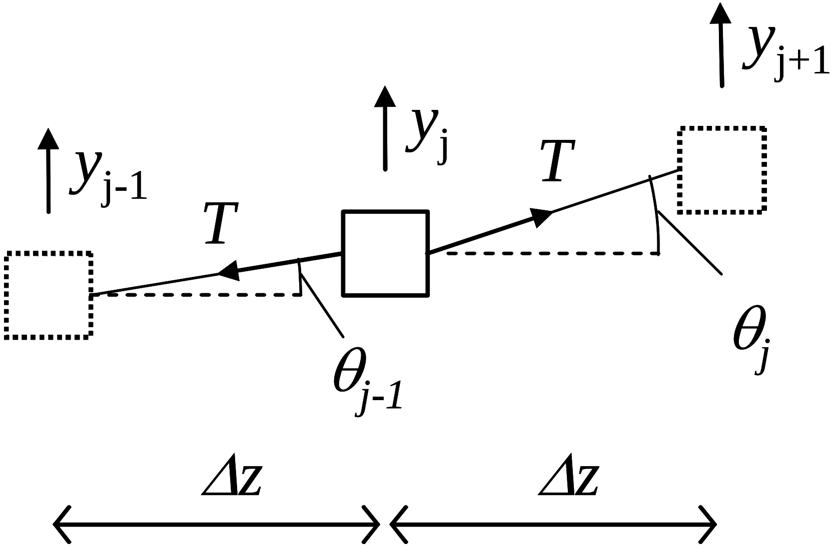
A force $T \sin\theta_j$ pulls mass j upwards

A force $T \sin\theta_{j-1}$ pulls mass j downwards

So the net upwards force on mass j is

$$F_j = T\left(\sin\theta_j - \sin\theta_{j-1}\right)$$

Classical wave equation



For small angles

$$\sin\theta_j \simeq \frac{y_{j+1}-y_j}{\Delta z}, \quad \sin\theta_{j-1} \simeq \frac{y_j - y_{j-1}}{\Delta z}$$

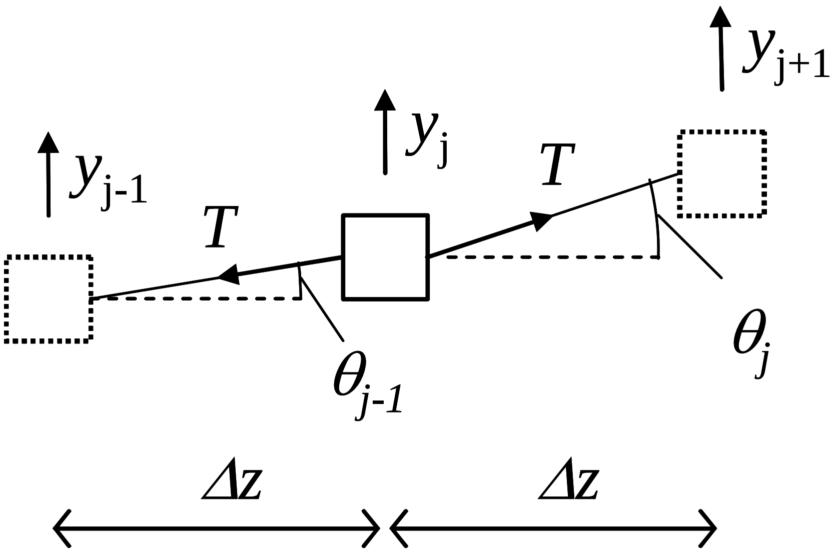
So $F_j = T\left(\sin\theta_j - \sin\theta_{j-1}\right)$

becomes

$$F_j \simeq T\left[\frac{y_{j+1}-y_j}{\Delta z} - \left(\frac{y_j - y_{j-1}}{\Delta z}\right)\right]$$

$$= T\left[\frac{y_{j+1}-2y_j+y_{j-1}}{\Delta z}\right]$$

Classical wave equation



In the limit of small Δz

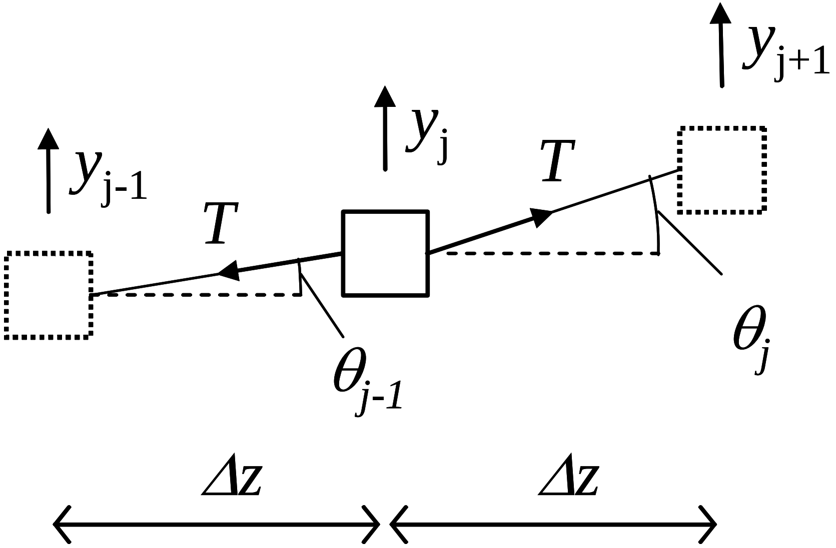
the force on the mass *j* is

$$F = T\left[\frac{y_{j+1} - 2y_j + y_{j-1}}{\Delta z}\right]$$

$$= T\Delta z\left[\frac{y_{j+1} - 2y_j + y_{j-1}}{(\Delta z)^2}\right]$$

$$= T\Delta z\frac{\partial^2 y}{\partial z^2}$$

Classical wave equation



Note that, with

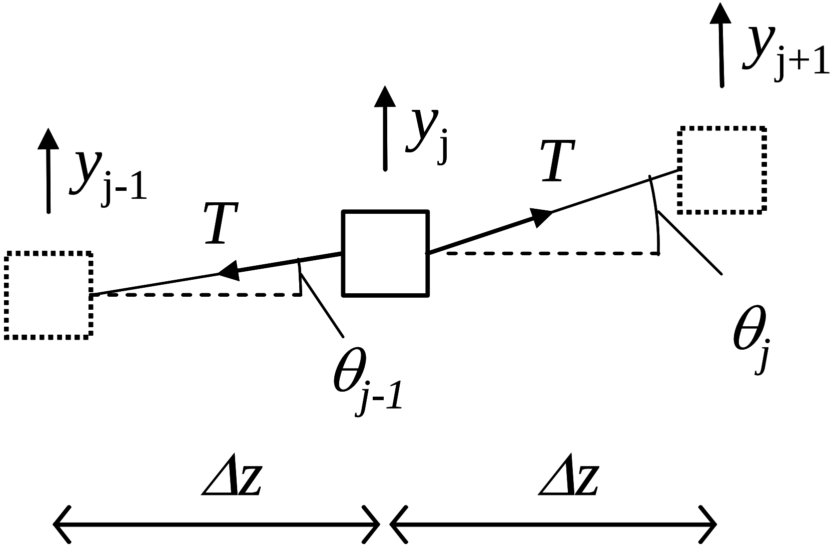
$$F = T\Delta z \frac{\partial^2 y}{\partial z^2}$$

we are saying that

the force F is proportional to the curvature of the "string" of masses

There is no net vertical force if the masses are in a straight line

Classical wave equation

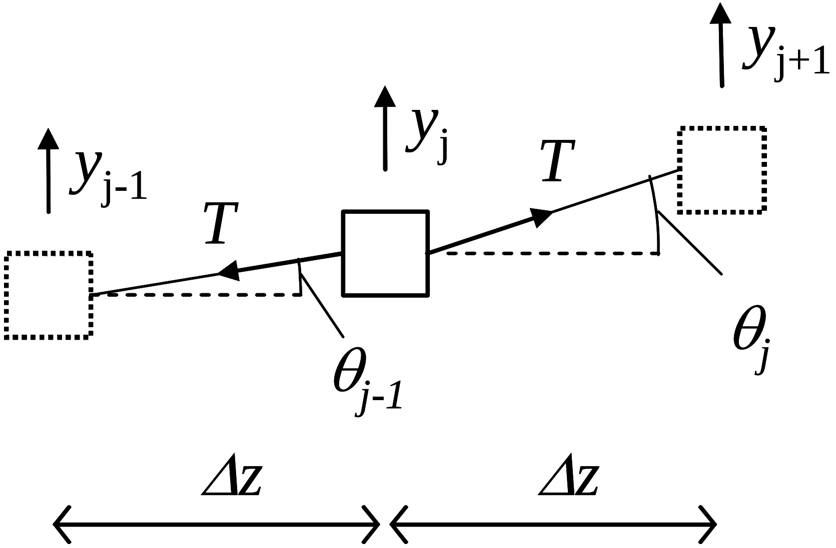


Think of the masses as

- the amount of mass per unit length in the z direction, that is
  - the linear mass density ρ times Δz, i.e., $m = \rho \Delta z$
    - Then Newton's second law gives

$$F = m \frac{\partial^2 y}{\partial t^2} = \rho \Delta z \frac{\partial^2 y}{\partial t^2}$$

Classical wave equation



Putting together

$$F = T\Delta z \frac{\partial^2 y}{\partial z^2} \text{ and } F = \rho \Delta z \frac{\partial^2 y}{\partial t^2}$$

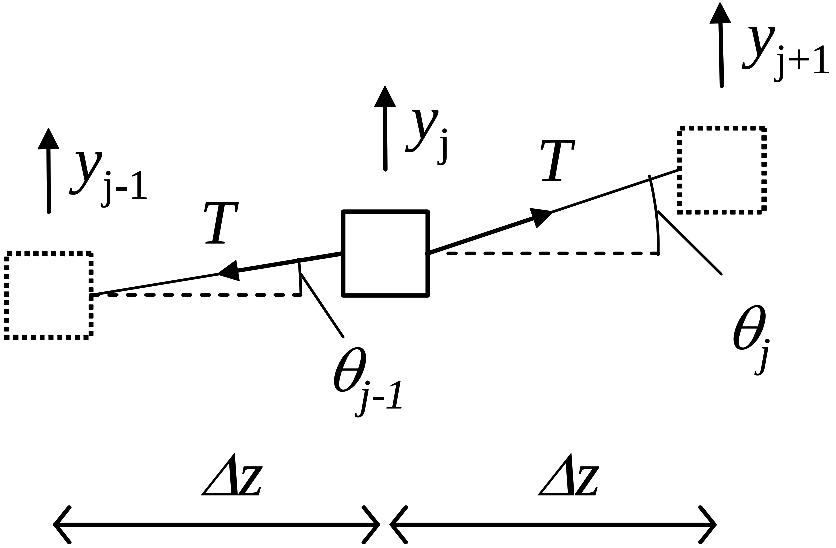
gives

$$T\Delta z \frac{\partial^2 y}{\partial z^2} = \rho \Delta z \frac{\partial^2 y}{\partial t^2}$$

i.e.,

$$\frac{\partial^2 y}{\partial z^2} = \frac{\rho}{T} \frac{\partial^2 y}{\partial t^2}$$

Classical wave equation



Rewriting

$$\frac{\partial^2 y}{\partial z^2} = \frac{\rho}{T}\frac{\partial^2 y}{\partial t^2}$$

with

$$v^2 = T / \rho$$

gives

$$\frac{\partial^2 y}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = 0$$

which is a wave equation for a wave with velocity $v = \sqrt{T / \rho}$

Wave equation solutions – forward waves

We remember that any function of the form $f\left(z - vt\right)$

is a solution of the wave equation

and is a wave moving to the right with velocity v

Wave equation solutions – forward waves

We remember that any function of the form $f(z - vt)$

is a solution of the wave equation

and is a wave moving to the right with velocity v

Wave equation solutions – backward waves

We remember that any function of the form $g\left(z+vt\right)$

is a solution of the wave equation

and is a wave moving to the left with velocity v

Oscillations and waves 2

The Helmholtz equation

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Monochromatic waves

Often we are interested in waves oscillating at one specific (angular) frequency ω

- monochromatic waves
  - i.e., temporal behavior of the form $\sin(\omega t)$ or $\cos(\omega t)$
    - or any combination of these

For any such combination

$$\frac{\partial^2 y}{\partial t^2} = -\omega^2 y$$

Monochromatic waves

For a monochromatic situation, with $\frac{\partial^2 y}{\partial t^2} = -\omega^2 y$

the wave equation $\frac{\partial^2 y}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = 0$

becomes $\frac{d^2 y(z)}{dz^2} + k^2 y(z) = 0$ where $k^2 = \frac{\omega^2}{v^2}$

We can think of this function $y(z)$

as being a "snapshot" of this monochromatic wave

so we take the "total" spatial derivative of this snapshot

giving a "simple" differential equation in one variable

The Helmholtz wave equation

This equation for a monochromatic wave

$$\frac{d^2 y(z)}{dz^2} + k^2 y(z) = 0$$

is called the "Helmholtz wave equation"

here given in its simplest, one-dimensional form

It is essentially the simplest useful wave equation we can construct

Mathematical form of the Helmholtz equation

This equation $\frac{d^2 y(z)}{dz^2} + k^2 y(z) = 0$

is also an eigen equation

with eigenvalue $-k^2$

though the Helmholtz equation is a differential equation in z, not t

Just as before for the simple harmonic oscillator

this equation can be written in the form $A\mathbf{y} = -k^2\mathbf{y}$

A corresponds now to the linear operator d^2/dz^2

Mathematical form of the Helmholtz equation

For the simple harmonic oscillator

the eigenvalue was fixed by the choice of the spring constant and the mass

For the Helmholtz equation for a string

we may have chosen a definite density ρ

and a definite tension T

But those only set the wave velocity magnitude v

which only sets the ratio of ω^2 and k^2, i.e., $v^2 = \dfrac{T}{\rho} = \dfrac{\omega^2}{k^2}$

so we can choose any frequency ω

but we have to make a choice, and that sets k

Mathematical form of the Helmholtz equation

We note that we can generalize the Helmholtz equation

from the "one-dimensional" form

to a "three-dimensional form"

$$\nabla^2 \psi(\mathbf{r}) + k^2 \psi(\mathbf{r}) = 0$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

We have to use partial derivatives here because we have three coordinate directions

Oscillations and waves 2

Standing waves

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General solution

Just as for the simple harmonic oscillator equation

for the Helmholtz equation

$$\frac{d^2y(z)}{dz^2} + k^2 y(z) = 0$$

we similarly have a general solution

$$y = A\sin kz + B\cos kz$$

where A and B can be any constants

Without further conditions

any such wave could be a solution

Standing waves

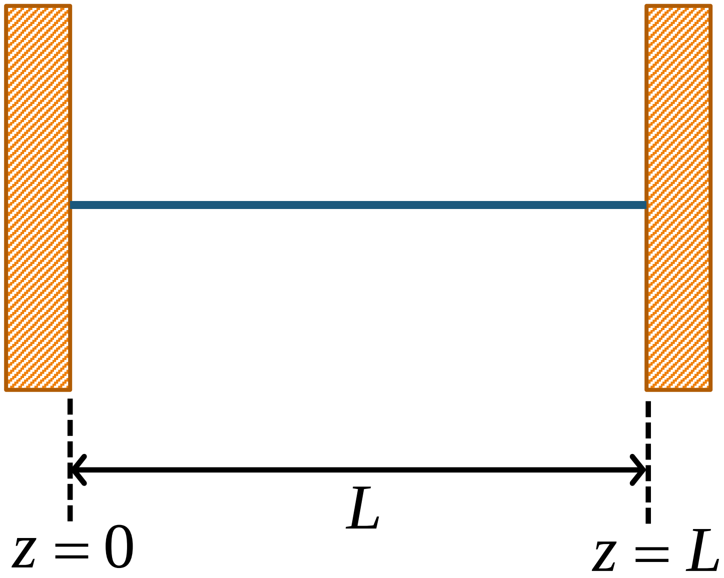
Now presume that the string is tied to rigid posts or walls at either end

- Now we are imposing boundary conditions

The wave has to be zero at the walls

- So we require

$$y(z) = 0 \text{ for } z = 0, L$$



Standing waves

Now we know that, at $z = 0$

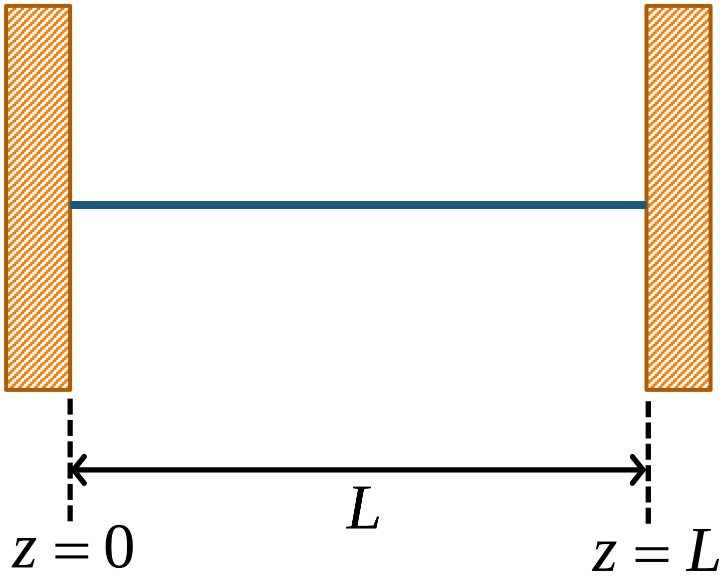
$$\sin kz \equiv \sin 0 = 0$$

and $$\cos kz \equiv \cos 0 = 1$$

So to have $y(0) = 0$ in a solution in

$$y = A\sin kz + B\cos kz$$

we require $B = 0$



Standing waves

Hence, given our boundary conditions

solutions are of the form

$$y(z) = A \sin kz$$

for some k and A

We also require that $y(L) = 0$

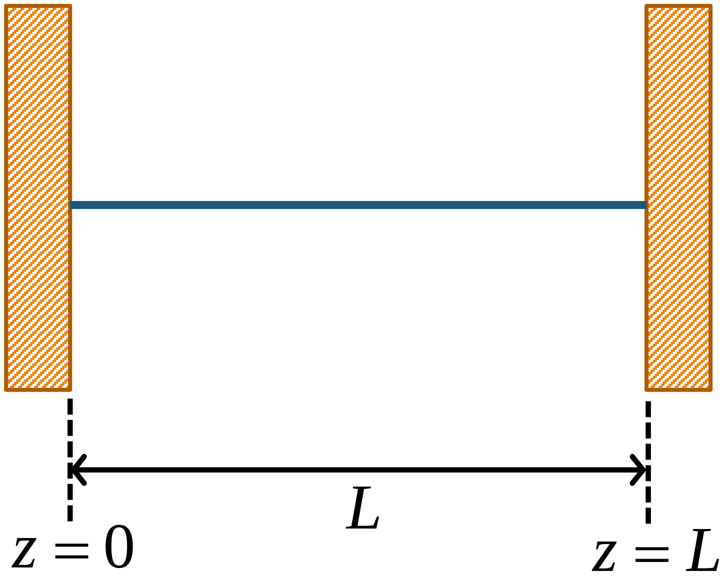
Now, $\sin\theta = 0$ requires that $\theta = n\pi$

where n is an integer

So k takes one of the values

$$k_n = n\pi / L$$

where n is an integer



Standing waves

So the eigenvalues are $k_n = n\pi / L$

so the resulting eigenfunctions are

$$y_n(z) = A\sin\left(\frac{n\pi z}{L}\right)$$

$n = 0$ is a trivial case

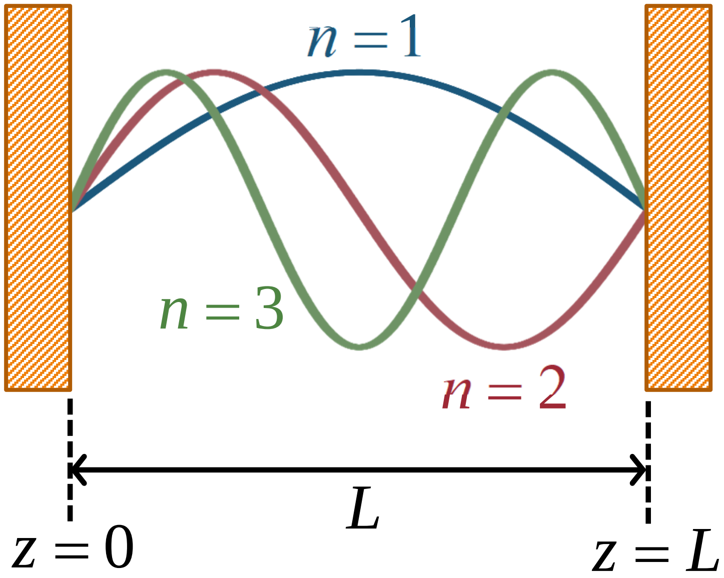
the wave would be 0 everywhere

Solutions for negative n are

essentially the same as for positive n

Hence, we set $n = 1, 2, 3, \ldots$.

The standing waves are the eigen functions or "eigenmodes"



Standing waves

For a density and tension, we know

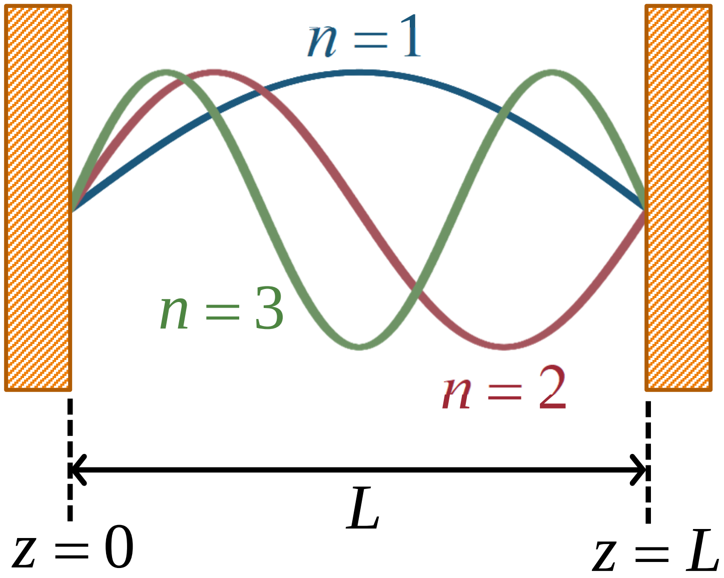
$$v^2 = \frac{T}{\rho} = \frac{\omega^2}{k^2}$$

or equivalently $\omega = vk$

So, with $k_n = n\pi / L$

we conclude the allowed (angular) frequencies are

$$\omega_n = vk_n = \frac{n\pi v}{L}$$



Standing waves

So, for monochromatic wave solutions

for a string between two rigid posts

"oscillating modes" only exist for specific (eigen) frequencies

which form a harmonic series with integer ratios

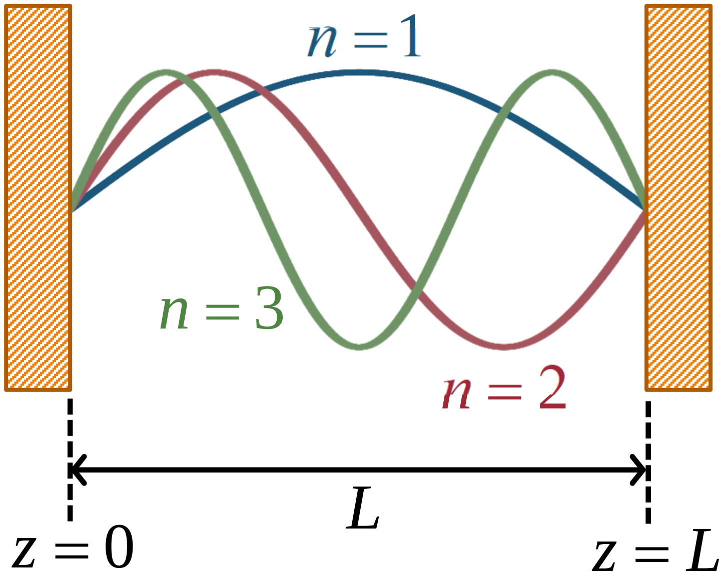
In two dimensions

such as cymbals or drum heads

or three dimensions

such as bells

frequencies may not be in integer ratios



Standing waves

We can also view standing waves

- as sums of forward and backward propagating waves

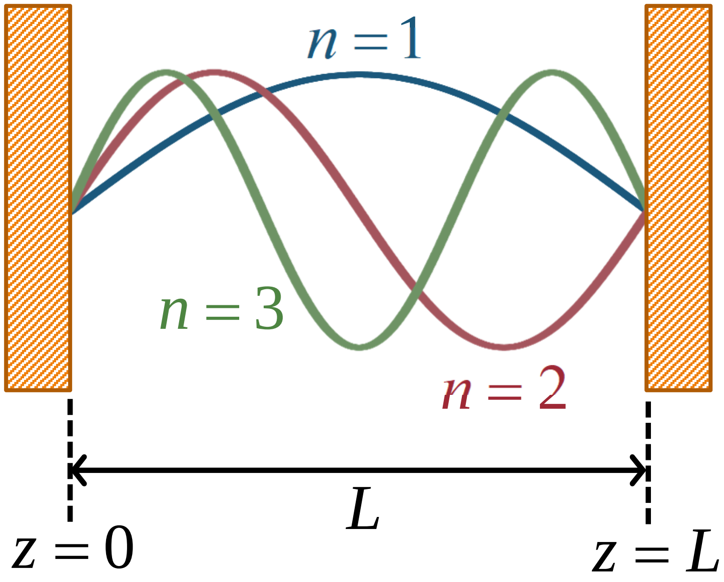
The "right going" $A\sin(kz - \omega t)$

and "left going" $A\sin(kz + \omega t)$

- waves are each solutions of the full wave equation

Because this wave equation is linear

- the sum or "superposition" of these two waves is also a solution



Standing waves

An equal combination of forward and backward waves, e.g.,

$$\Psi(z,t) = A\sin(kz - \omega t) + A\sin(kz + \omega t)$$

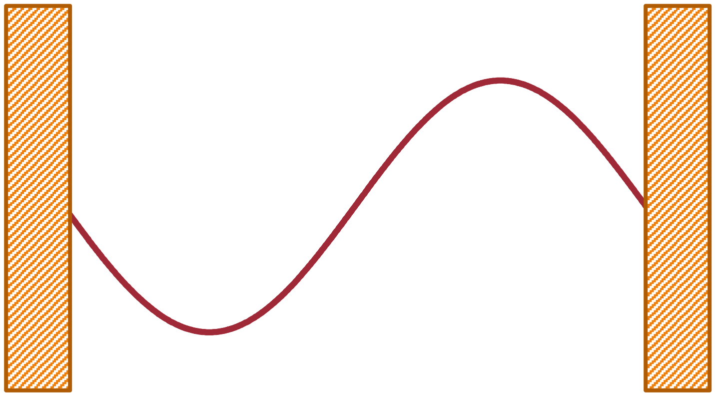
$$= 2A\cos\omega t \sin(kz)$$

where $k = \omega / v$

gives "standing waves"

E.g., for a rope tied to two walls a distance L apart

with $k = 2\pi / L$ and $\omega = 2\pi v / L$

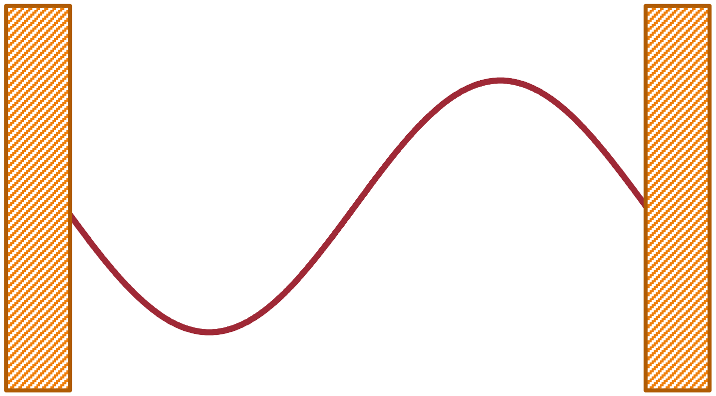


Standing waves

We can also think of

the right-propagating wave
reflecting off the right barrier
to give the left-propagating wave, and

the left-propagating wave
reflecting off the left barrier
to give the right-propagating wave



Standing waves

When a wave reflects off a "hard wall"

- the reflected wave is minus the incident wave
  - so the sum always equals zero at the position of the barrier
    - as required for the net wave to be zero at the barrier

A wave reflecting from a hard wall

- has its phase changed by 180 degrees