Oscillations and waves 3

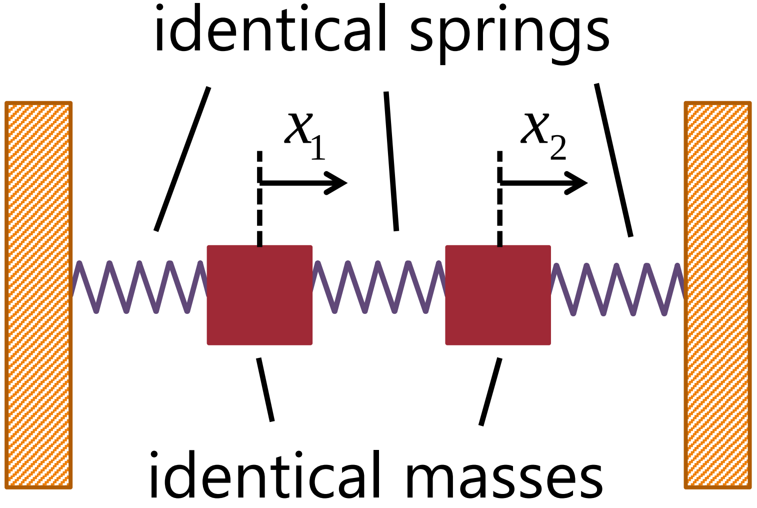
A coupled oscillator

Modern physics for engineers

David Miller

Coupled oscillator

- Suppose we have two masses and three springs
  - The masses can only move side to side
  - and their movement is frictionless
- This is like two mass-on-a-spring oscillators
  - coupled by an additional spring



Coupled oscillator

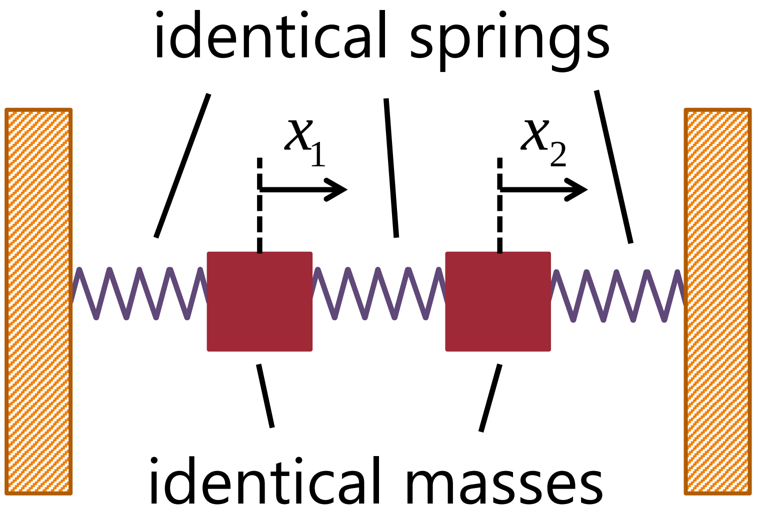
We take the two masses to be equal

- of value m

We take all springs to have the same spring constant K

At equilibrium

- we presume no net stretching or compression of any springs



Coupled oscillator

The position of the left mass

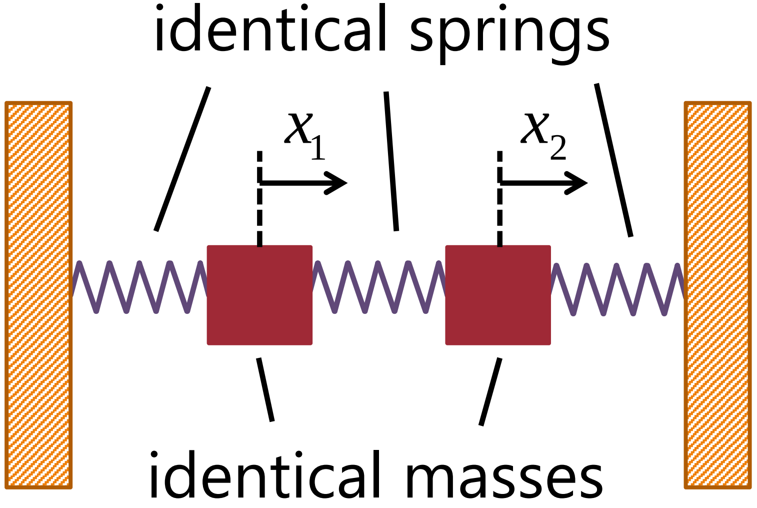
relative to the equilibrium position

is x_1

Similarly the position of the right mass

relative to the equilibrium position

is x_2



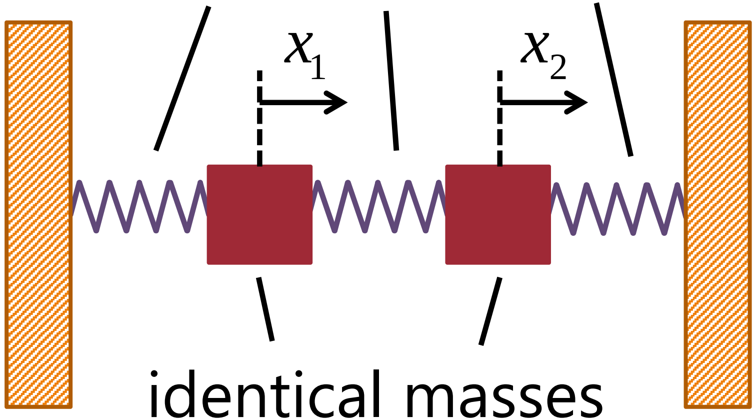
Coupled oscillator

From the stretching of the left spring

the restoring force pulling the left mass to the left

back to the equilibrium position

is $-Kx_1$



Coupled oscillator

There is also a force on the left mass from the middle spring

The middle spring is stretched by an amount $x_2 - x_1$

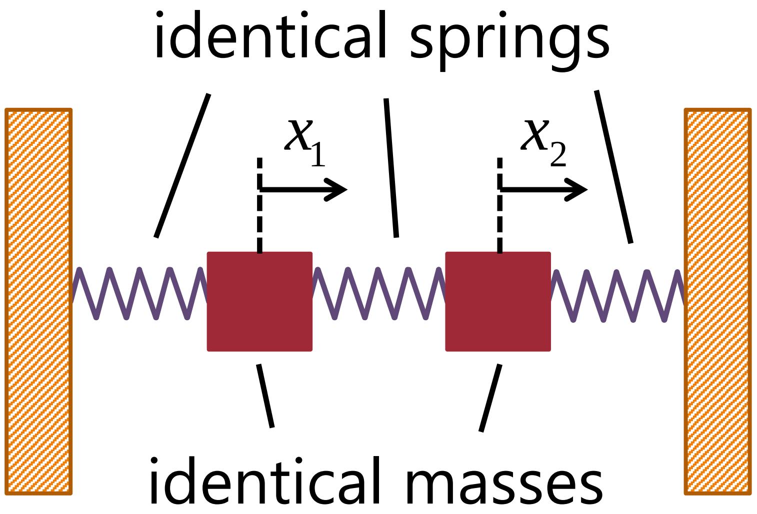
giving a force on the left mass

$$K\left(x_2 - x_1\right)$$

pulling it to the right

so the net force to the right on the left mass is

$$-Kx_1 + K\left(x_2 - x_1\right) = -2Kx_1 + Kx_2$$



Coupled oscillator

Applying Newton's second law to the left mass gives

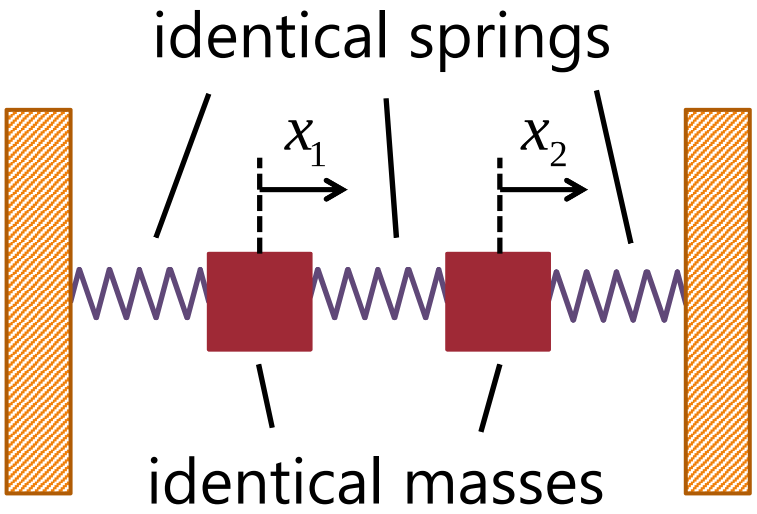
$$m\frac{d^2x_1}{dt^2} = -2Kx_1 + Kx_2$$

A similar analysis for the right mass gives

$$m\frac{d^2x_2}{dt^2} = -K\left(x_2 - x_1\right) - Kx_2$$

$$= -2Kx_2 + Kx_1$$

We need to solve these coupled equations



Finding the eigenmodes

Finding the eigenmodes

To look for eigenmodes

we presume that everything that is oscillating is oscillating at the same frequency

which we now presume to be some (angular) frequency ω

and we look for solutions

For any solutions oscillating in the form $\sin \omega t, \cos \omega t$

or any linear combination

we can replace $\frac{d^2}{dt^2}$ with $-\omega^2$

Finding the eigenmodes

Hence the equations

$$m\frac{d^2x_1}{dt^2} = -2Kx_1 + Kx_2 \qquad m\frac{d^2x_2}{dt^2} = -2Kx_2 + Kx_1$$

become

$$-\omega^2 m x_1 = -2Kx_1 + Kx_2 \qquad -\omega^2 m x_2 = -2Kx_2 + Kx_1$$

which we can rewrite in matrix form as

$$\begin{bmatrix} 2K & -K \\ -K & 2K \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \omega^2 m \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The top line is the first equation

and the bottom line is the second equation

Finding the eigenmodes

If we now choose to write for some variable λ, $\lambda = \frac{\omega^2 m}{K}$

then $\begin{bmatrix} 2K & -K \\ -K & 2K \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \omega^2 m \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

becomes $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

which is, in abstract mathematical notation, $\mathrm{A}\mathbf{x} = \lambda \mathbf{x}$

with $\mathrm{A} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ $\quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Finding the eigenmodes

Hence we have reduced this problem to a matrix eigenvalue and eigenvector problem

We want the eigenvalues λ

for which $\mathrm{A}\mathbf{x} = \lambda\mathbf{x}$ has solutions

and we want the eigenvectors $\mathbf{x}$ corresponding to each eigenvalue λ

Finding the eigenmodes

Rewriting the equation $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

as $\begin{bmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

we know from matrix algebra that

this only has a solution if the determinant of the matrix is zero

i.e., $(2-\lambda)(2-\lambda)-1=0$

i.e., $\lambda^2 - 4\lambda + 3 = 0$

Finding the eigenmodes

Solving this quadratic $\lambda^2 - 4\lambda + 3 = 0$

gives $\lambda = 2\left(1 \pm \frac{1}{2}\right)$ or equivalently $\omega^2 = \frac{2K}{m}\left(1 \pm \frac{1}{2}\right)$

i.e., $\lambda = 1$ or equivalently $\omega = \sqrt{K/m}$

and $\lambda = 3$ or equivalently $\omega = \sqrt{3K/m}$

Substituting each value of λ back into $\begin{bmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

lets us solve for the eigenvectors in each case

Finding the eigenmodes

So finally we have

for $\lambda = 1$ ($\omega = \sqrt{K/m}$), $\mathbf{x} \propto \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

i.e., both masses moving in the same direction

for $\lambda = 3$ ($\omega = \sqrt{3K/m}$), $\mathbf{x} \propto \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

i.e. both masses moving in the opposite direction

Note the higher frequency of the "opposite direction" mode

Oscillations and waves 3

Inner products, orthogonality, and basis sets

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Mathematical properties of modes

The modes we have looked at so far

- describe oscillations at specific frequencies

In linear systems

- the mathematics behind modes is quite generally useful
  - well beyond simple oscillations

Mathematical properties of modes

We can look further at this mathematics

- and its use in describing linear systems

We use some results from the coupled oscillator

- to illustrate these mathematical properties

Inner products and orthogonality

Inner products

We can take the "dot" product between two geometrical vectors

for example, in a two-dimensional x-y plane

with unit vectors $\mathbf{i}$ and $\mathbf{j}$ in the x and y directions

For $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{b} = -5\mathbf{i} + 4\mathbf{j}$

remembering the dot product is the sum of the products of the components

$$\mathbf{a} \cdot \mathbf{b} = 2 \times (-5) + 3 \times 4 = 2$$

Inner products

We can multiply two vectors in matrix-vector notation

provided the "dimensions" of the vectors match

We can multiply a column vector on the "right"

and a row vector on the "left"

if the column and the row are the same "length"

Writing **a** as a row vector and **b** as a column vector

we would have

$$\mathbf{a}\cdot\mathbf{b} = \begin{bmatrix} 2 & 3 \end{bmatrix}\begin{bmatrix} -5 \\ 4 \end{bmatrix} = 2\times(-5) + 3\times 4 = 2$$

Inner products and orthogonality

If two geometrical vectors are at right angles

such as $\mathbf{c} = 2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{d} = -3\mathbf{i} + 2\mathbf{j}$

then their dot product is zero

$$\mathbf{c} \cdot \mathbf{d} = 2 \times (-3) + 3 \times 2 = 0$$

Generally, such products of row and column vectors

even if they are not geometrical vectors

can be called "inner products"

If the inner product of two non-zero vectors is zero

the vectors are "orthogonal"

a generalization of vectors being at right angles

Orthogonality of vectors

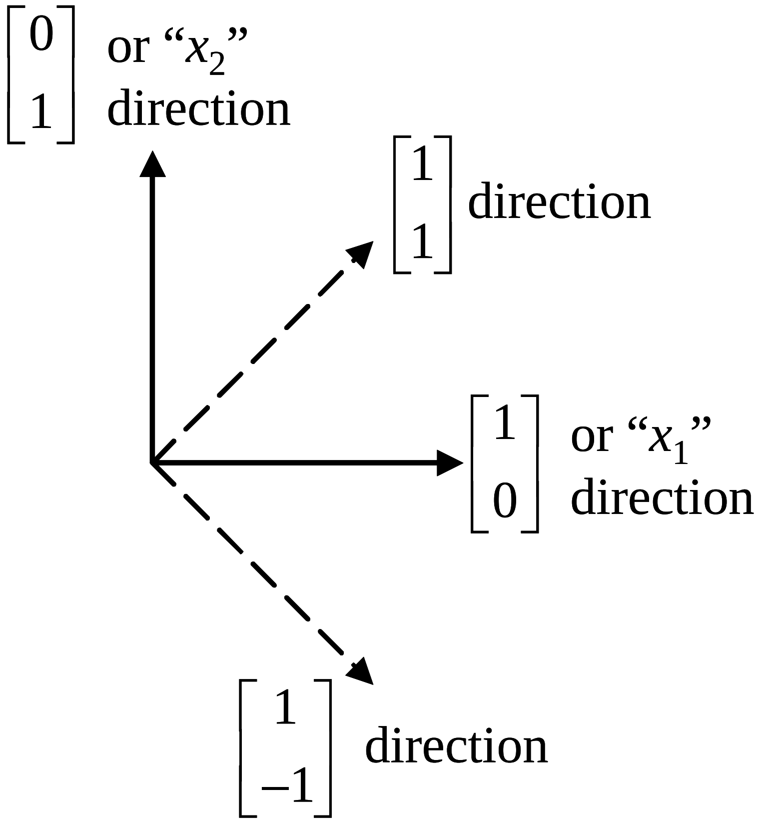
Note the coupled-oscillator eigenvectors

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are orthogonal

That is, $\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$

If we draw out these eigenvectors on a plane

they are geometrically orthogonal as we expect



Orthogonality of vectors

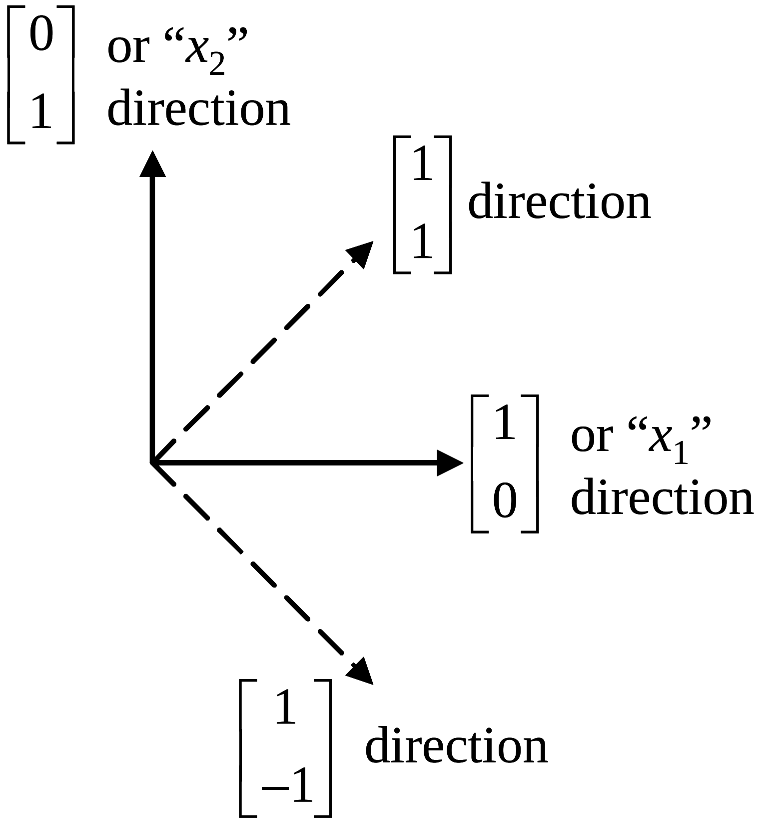
This orthogonality is not accidental

For a very broad class of linear physical problems

including quantum mechanical ones

the eigenfunctions or (eigen) modes

are orthogonal in this sense



Functions as vectors

Functions as vectors

- We are now thinking of a function as a vector in a mathematical space
  - Any pair of positions of the coupled oscillator masses
    - can be represented as a vector on a plane

Functions as vectors

The function here is a list of two numbers

- corresponding to the position of mass 1
- and the position of mass 2

We can represent any function as a list of numbers

- representing the results when mapping from some known list of values of the “argument”

Completeness and basis sets

Completeness and basis sets

These eigenfunctions or eigenmodes can be used

to describe any particular position of the two masses

This property that sets of modes can have

is called "completeness"

Any particular pair of positions of the two masses

$x_1 = f$ and $x_2 = g$

can be represented as $\begin{bmatrix} f \\ g \end{bmatrix} = f \begin{bmatrix} 1 \\ 0 \end{bmatrix} + g \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

or as $\begin{bmatrix} f \\ g \end{bmatrix} = \frac{(f+g)}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{(f-g)}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Completeness and basis sets

Because any such pair of positions $x_1 = f$ and $x_2 = g$

can be represented using combinations

of the (eigen) vectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

we can say this set of eigenvectors is complete

for representing any possible pair of positions

of the two masses

Completeness and basis sets

If a set of vectors can make up any vector in the "space" of interest

- here a 2-dimensional "plane"
  - we can call that set a "basis set" of vectors

If those basis vectors are all orthogonal to one another

- then we call the set an "orthogonal basis set"
  - or often just an "orthogonal basis"

Completeness and basis sets

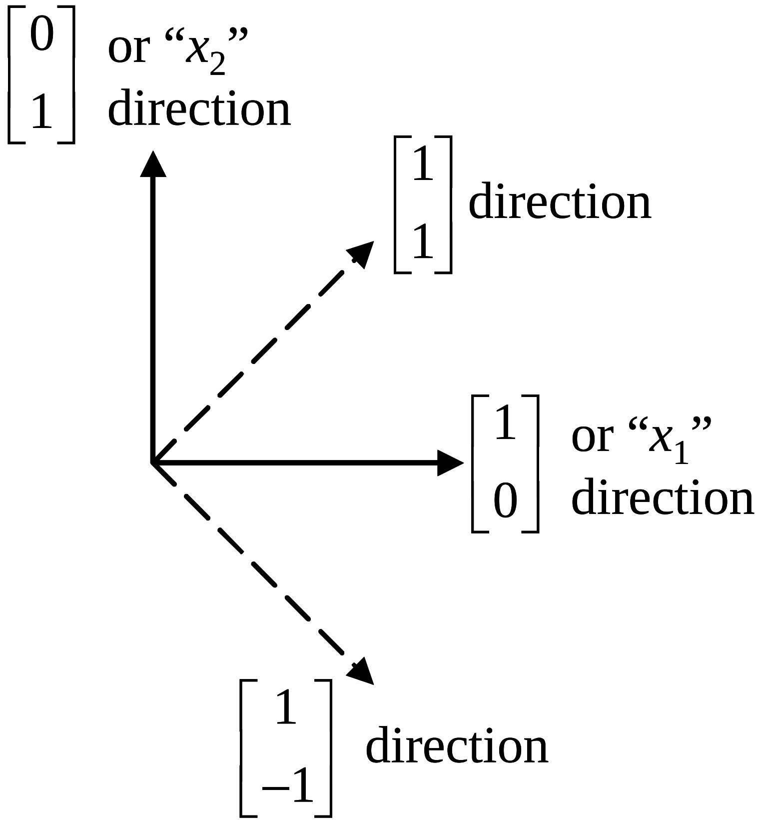
We could represent any point in the plane

in terms of its coordinates

along $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\-1\end{bmatrix}$ directions

or along $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ directions

Either pair of vectors is a complete orthogonal basis



Oscillations and waves 3

Hermitian operators and sets of functions

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Hermitian adjoints

Hermitian adjoints

Taking a Hermitian adjoint of a matrix involves

reflecting a matrix along its leading diagonal

i.e., taking the transpose of the matrix

and taking the complex conjugate of the elements

The Hermitian adjoint is often denoted with the "dagger" symbol "†"

E.g., for a 2 × 2 matrix, we have $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{\dagger} \equiv \begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix}$

Hermitian adjoints

We can also have Hermitian adjoints of matrices that are not square

- such as a vector

In the case of a 2-element vector, for example, we have

$$\begin{bmatrix} a \\ b \end{bmatrix}^\dagger \equiv \begin{bmatrix} a^* & b^* \end{bmatrix}$$

Again, we can think of this operation as

- reflecting about a “45°” diagonal line
  - i.e., from top left to bottom right
    - and taking the complex conjugate of the elements

Hermitian adjoints

We can think of the Hermitian adjoint as

- like the idea of a complex conjugate
  - but now generalized to matrices or operators

A simple number can be thought of

- as a 1×1 matrix
  - and the Hermitian adjoint of that "matrix"
    - is simply the complex conjugate of the number

Inner product

Now that we may have complex numbers

we need to clarify the definition of the inner product

We must use the Hermitian adjoint vector as the one on the left

i.e., the inner product of $\begin{bmatrix} a \\ b \end{bmatrix}$ and $\begin{bmatrix} c \\ d \end{bmatrix}$

is given by $\begin{bmatrix} a^* & b^* \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = a^* c + b^* d$

Hermitian matrices and operators

Hermitian matrices and operators

A Hermitian matrix is

- one that is equal to its own Hermitian adjoint
  - In abstract operator notation
    - a Hermitian operator is one for which

$$\hat{A}^{\dagger} = \hat{A}$$

Hermitian matrices

These three matrices are Hermitian

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & i \\ -i & -2 \end{bmatrix} \quad \begin{bmatrix} 0 & 1-i & -3 \\ 1+i & -1 & \exp\left(i/\sqrt{2}\right) \\ -3 & \exp\left(-i/\sqrt{2}\right) & 1 \end{bmatrix}$$

but these three are not

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix} \quad \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$$

Hermitian operators and physical problems

Hermitian operators and physical problems

For a large number of linear physical problems

e.g., for frictionless or loss-less systems

the key linear operators are Hermitian

and can be represented by Hermitian matrices

Hermitian operators and physical problems

A large number of the operators in quantum mechanics are Hermitian

The eigenfunctions of those Hermitian operators are orthogonal

with real eigenvalues

giving a complete basis set for the relevant space

Hermitian operators and physical problems

Not only are eigenfunctions and eigenmodes interesting for the physical behaviors they describe

they can also have remarkable and useful mathematical properties

specifically

- orthogonality
- completeness

so we can use them to describe many behaviors of physical systems

Standing waves and Fourier series

Standing waves and Fourier series

For the standing wave modes on a string

they should be a complete set for describing

any function of position between 0 and L

(and having the value zero at positions 0 and L)

In fact, we know they are complete

because these functions can form a Fourier series

which in this case we could write as

$$f(z) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi z}{L}\right)$$

where the a_n are appropriate numbers (coefficients)

Standing waves and Fourier series

In such a Fourier series representation of a function

$$f(z) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi z}{L}\right)$$

- the set of coefficients a_n is just as good a way of describing the function
  - as the list of values of the function for all values of z of interest

Our modes here for standing waves on a string

- are identical to the sine waves used in the Fourier series

Generality of basis sets of functions

Generality of basis sets of functions

Such basis sets of functions

can be used to represent any function

even where the physical problem

is different from the one used in deriving the set

Generality of basis sets of functions

For example

- we could have a string whose density varied along its length
  - The modes of such a string would not be simple sine waves
    - but, at any given time, we could represent the shape of the string
      - as a sum of our sine wave "modes" or functions

Generality of basis sets of functions

Here, we are using the mathematical properties of

- the set of eigenfunctions of an operator
  - even though that operator may not be the one that corresponds to the current physical problem

Generality of basis sets of functions

There is an infinite number of possible basis sets of functions to describe functions in any given space

- For a given problem
  - the right choice of basis makes the problem simpler to solve

This notion of multiple different possible basis sets

- is central to the mathematics of quantum mechanics

Modes as the eigenfunctions of operators

Modes as eigenfunctions of operators

The broadest possible definition we can have for a mode is

a mode is an eigenfunction of an operator that describes a physical system

Modes as eigenfunctions of operators

For linear physical systems

that can be described by a Hermitian operator that

can be approximated to any degree of accuracy by a finite matrix

- the modes are the eigenfunctions of the operator
- they are orthogonal
- they are complete, and
- they have real eigenvalues

Modes as eigenfunctions of operators

For oscillating modes we can say

- in an oscillating mode
  - everything that is oscillating
    - is oscillating at the same frequency