The quantum view of the world 5

Tunneling

Modern physics for engineers

David Miller

Tunneling

In quantum mechanics

- a particle can get to the other side of a "hill"
  - even though it does not have enough energy to get over the "hill"

This is "tunneling"

Tunneling

Tunneling is common in modern electronics, e.g.,

- in "flash" memory
  - which is written and erased by tunneling through dielectrics
- in transistors
  - where there is undesired tunneling through the gate oxide

Tunneling in optics

Tunneling in optics

- Shining a flashlight from underwater
  - will lead to total internal reflection past a certain angle
- But the light “tunnels” a short distance into the air

Tunneling in optics

Putting a piece of glass close to the surface

- but not touching it
  - can allow light to "tunnel" from the water into the glass
    - known as "frustrated total internal reflection"
      - a well-known classical wave phenomenon

A potential barrier of finite height

Nature of solutions for a finite barrier

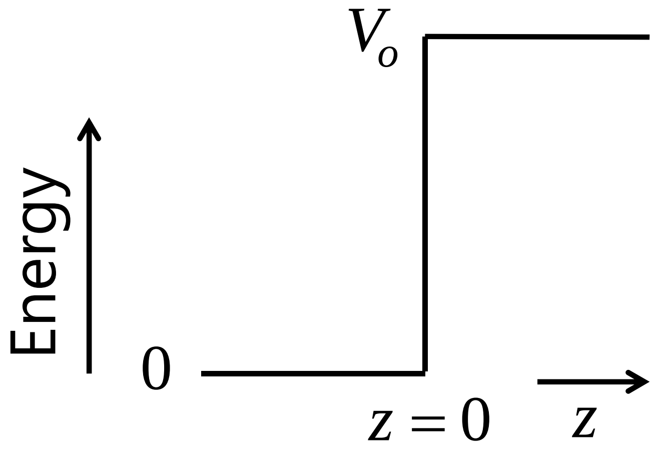
Infinitely thick barrier

Suppose we have a barrier of height V_o

- with potential 0 to the left of the barrier

A quantum mechanical wave is incident from the left

- The energy E of this wave is positive
  - i.e., $E > 0$



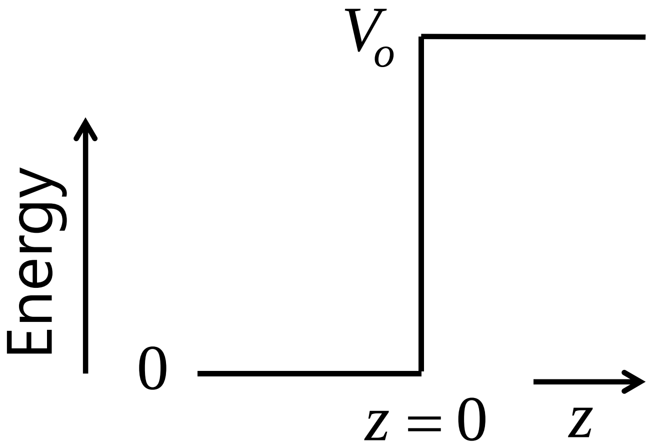
Infinitely thick barrier

We allow for reflection from the barrier
  into the region on the left

Using the general solution on the left
  with complex exponential waves

$$\psi_{left}(z) = A\exp(ikz) + B\exp(-ikz)$$

where, as before $k = \sqrt{2mE/\hbar^2}$



$A\exp(ikz)$ is the incident wave, going right
$B\exp(-ikz)$ is the reflected wave, going left

Infinitely thick barrier

Presume that $E < V_o$

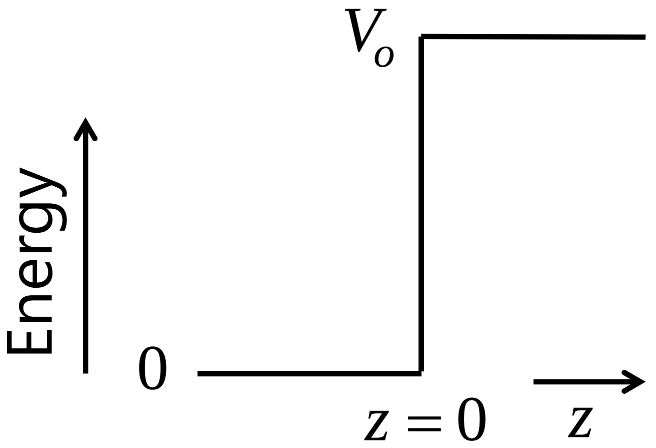
i.e., the incident wave energy is less than the barrier height

Inside the barrier, the wave equation is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(z)}{dz^2} + V_o\psi(z) = E\psi(z)$$

i.e., mathematically

$$\frac{d^2\psi(z)}{dz^2} = \frac{2m}{\hbar^2}(V_o - E)\psi(z)$$



Infinitely thick barrier

The general solution of

$$\frac{d^2\psi(z)}{dz^2} = \frac{2m}{\hbar^2}(V_o - E)\psi(z)$$

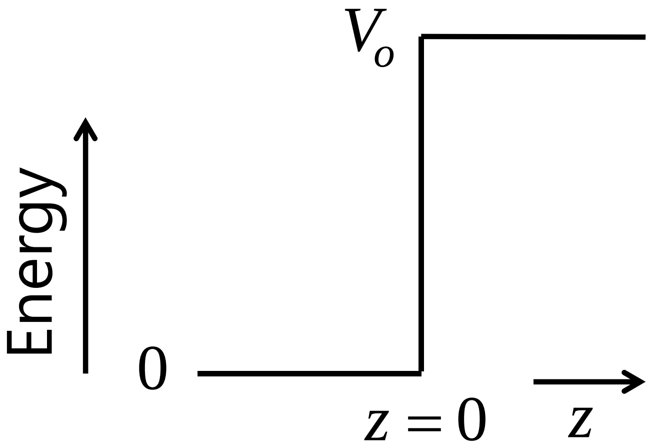
for the wave on the right is

$$\psi_{right}(z) = C\exp(-\kappa z) + D\exp(\kappa z)$$

where $\kappa = \sqrt{2m(V_o - E)/\hbar^2}$

We presume $D = 0$

otherwise the wave increases exponentially to the right for ever



Infinitely thick barrier

Hence the wave on the right

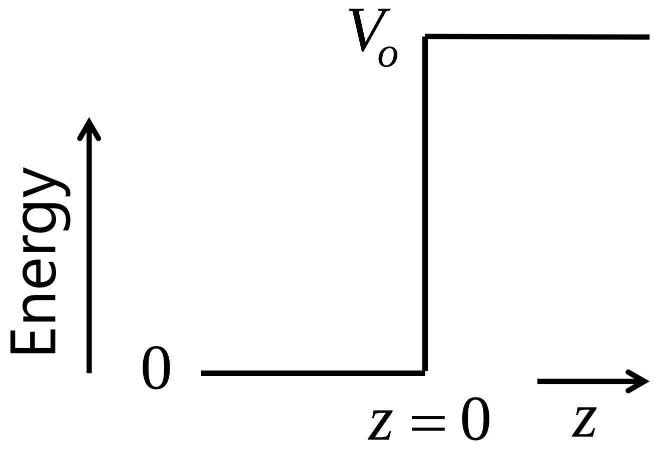
inside the barrier, is

$$\psi_{right}(z) = C\exp(-\kappa z)$$

with $\kappa = \sqrt{2m(V_o - E)/\hbar^2}$

This solution proposes that the wave inside the barrier is not zero

Instead, it falls off exponentially



The quantum view of the world 5

Solving for barriers of finite height

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A potential barrier of finite height

Boundary conditions

Boundary conditions

For our Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(z)}{dz^2}+V(z)\psi(z)=E\psi(z)$$

if we presume that E, V and ψ are finite

then $d^2\psi/dz^2$ must be finite also, so

$d\psi/dz$ must be continuous

If there was a jump in $d\psi/dz$

then $d^2\psi/dz^2$ would be infinite at that point

Boundary conditions

Also

$d\psi / dz$ must be finite

otherwise $d^2\psi / dz^2$ could be infinite

being the limit of a difference

involving infinite quantities

For $d\psi / dz$ to be finite

ψ must be continuous

Boundary conditions

Now that we have these two boundary conditions

ψ must be continuous

$d\psi / dz$ must be continuous

we can proceed to solve problems with finite "heights" of boundaries

A potential barrier of finite height

Solutions for a barrier of finite height

Infinitely thick barrier

Using the boundary conditions

we complete the solution

On the left, we have $\psi_{left}(z) = A\exp(ikz) + B\exp(-ikz)$

On the right we have $\psi_{right}(z) = C\exp(-\kappa z)$

Continuity of the wavefunction at $z=0$ gives $A + B = C$

Continuity of the wavefunction derivative at $z=0$ gives $ikA - ikB = -\kappa C$

i.e., $A - B = \frac{i\kappa}{k}C$

Infinitely thick barrier

Adding

$$A + B = C$$

$$A - B = \frac{i\kappa}{k} C$$

gives

$$2A = \left(1 + \frac{i\kappa}{k}\right) C = \left(\frac{k + i\kappa}{k}\right) C$$

Equivalently

$$C = \frac{2k}{k + i\kappa} A = \frac{2k\left(k - i\kappa\right)}{k^2 + \kappa^2} A$$

so we have found the amplitude C of the wave in the barrier in terms of the amplitude A of the incident wave

Infinitely thick barrier

Subtracting

$$A + B = C$$

$$A - B = \frac{i\kappa}{k} C$$

gives a similar relation between B and C

and we can deduce a relation

between A and B

So we can solve the entire problem here

leaving only one arbitrary overall constant

Probability densities

Note that we have to take the modulus squared of the entire wavefunction

so on the left we have

$$\left|\psi_{left}(z)\right|^2 = \left|A\exp(ikz) + B\exp(-ikz)\right|^2$$

and on the right

$$\left|\psi_{right}(z)\right|^2 = \left|C\exp(-\kappa z)\right|^2 = |C|^2\exp(-2\kappa z)$$

Note the probability density decays by 1/e

in a distance $1/2\kappa$

Example numbers

For a barrier of height $V_o = 2$ eV

and an incident electron of energy $E = 1.5$ eV

On the left $k \simeq 6.275 \times 10^9$ m^{-1} $\equiv 6.275$ nm^{-1}

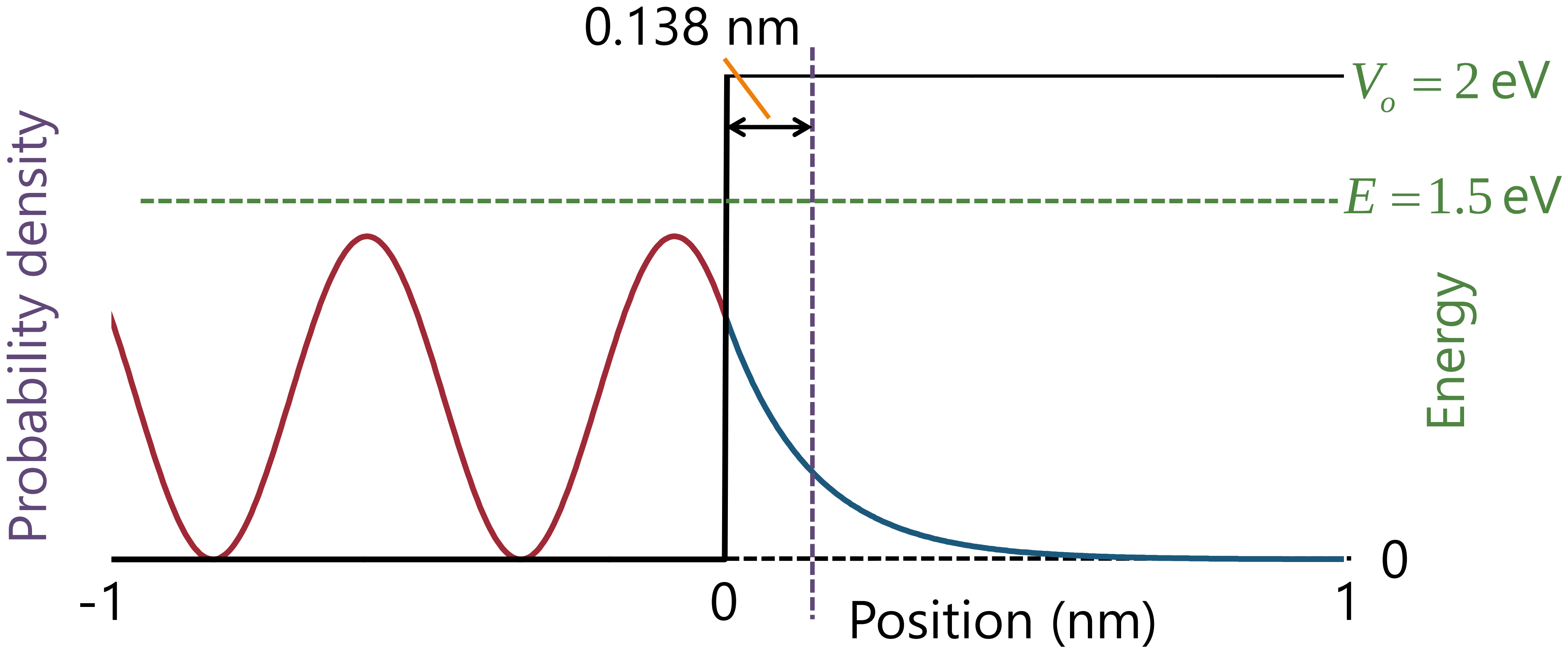
which corresponds to a wavelength of $\lambda \simeq 1.001$ nm

On the right $\kappa \simeq 3.623 \times 10^9$ m^{-1} $\equiv 3.623$ nm^{-1}

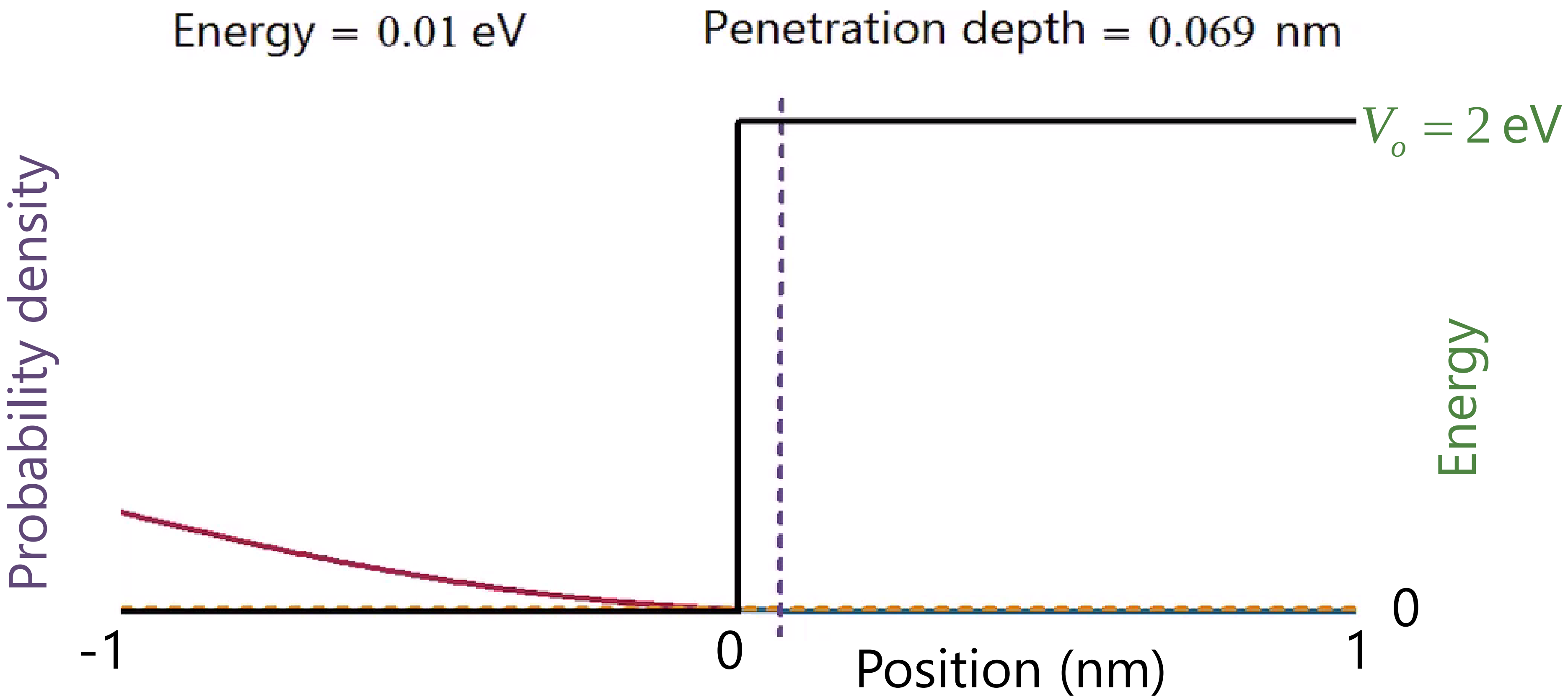
so the $1/e$ decay length of the probability density on the right is $1/2\kappa \simeq 0.138$ nm

Example probability density calculation

Note the standing wave and the phase change on reflection



Example probability density calculation



The quantum view of the world 5

Tunneling through a barrier

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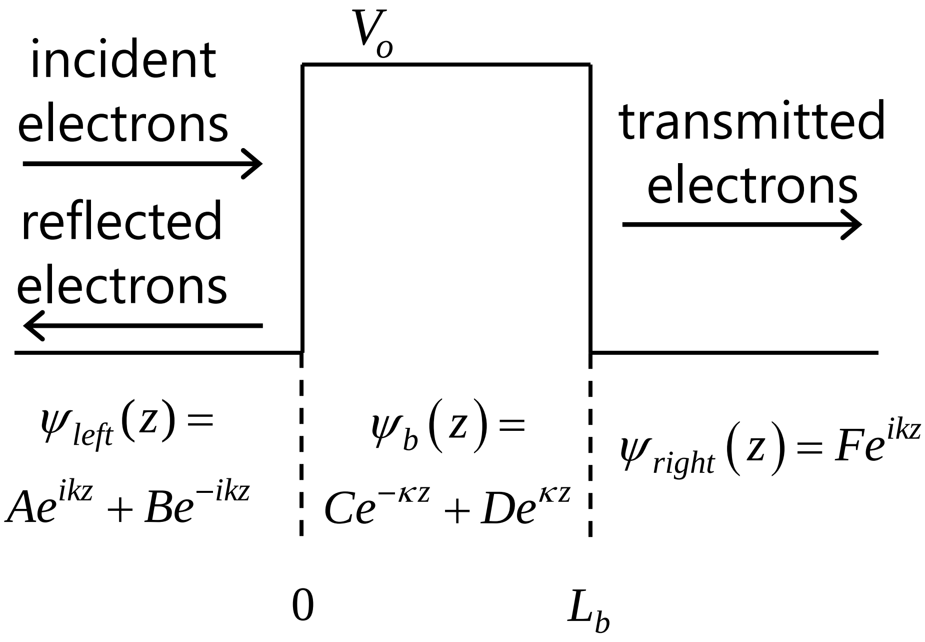
A potential barrier of finite height

A barrier of finite height and thickness

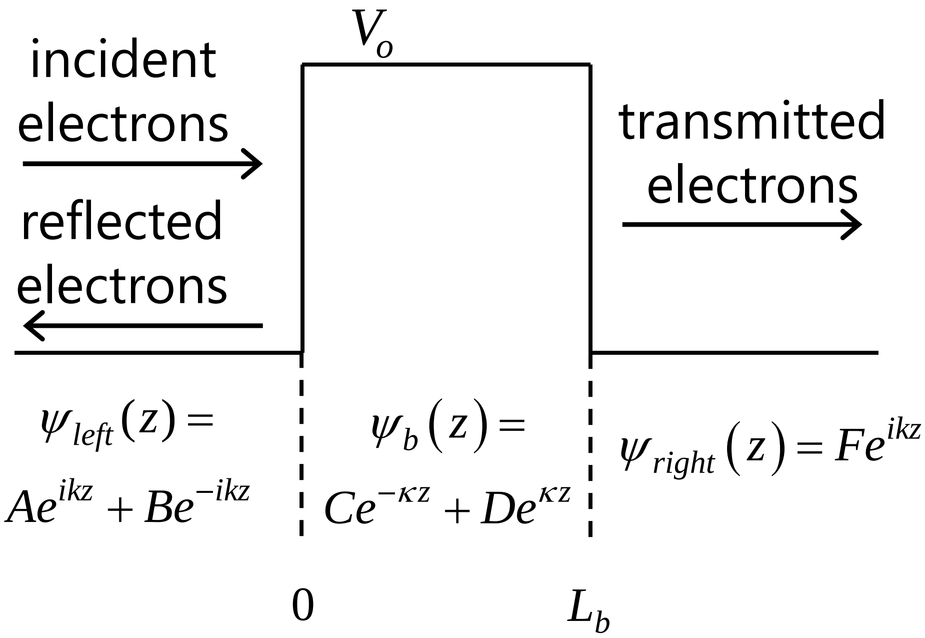
Tunneling through a barrier

- Consider a barrier of finite thickness, L_b
  - still with incident electron energy $E < V_o$
    - where V_o is the barrier height
- We presume an incident electron wave from the left
  - but none from the right



Tunneling through a barrier

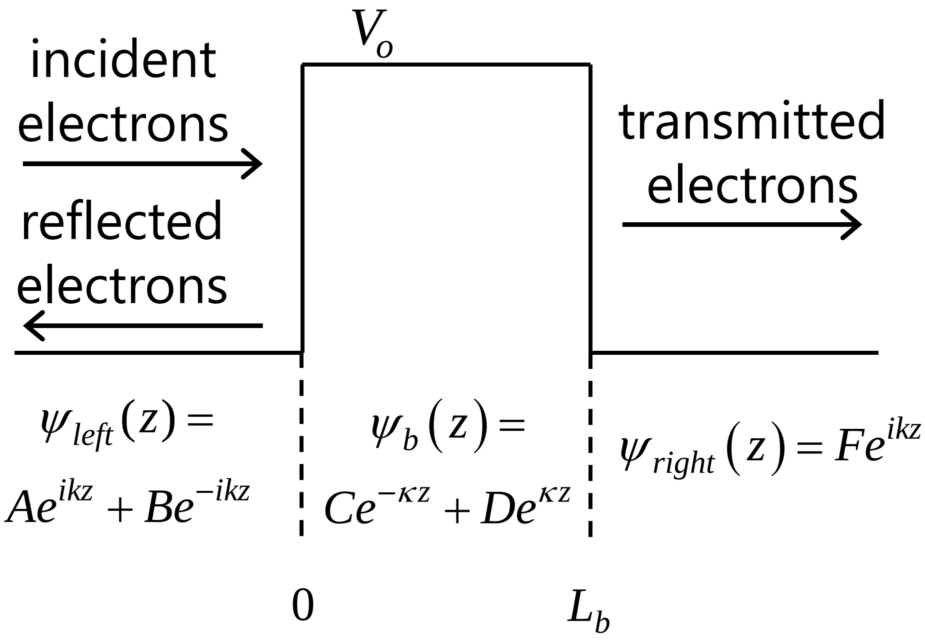
- Now we need to retain both exponentials in the barrier
  - The "growing" one
    - corresponds to a decaying one
    - from the "reflection" at the right side of the barrier



Tunneling through a barrier

We can solve this starting from the right

- Choose an arbitrary amplitude F
  - Deduce relations between C, D, and F using boundary conditions
  - Deduce relations between A, B, C, and D using boundary conditions



Tunneling through a barrier

Now the fraction of the incident current

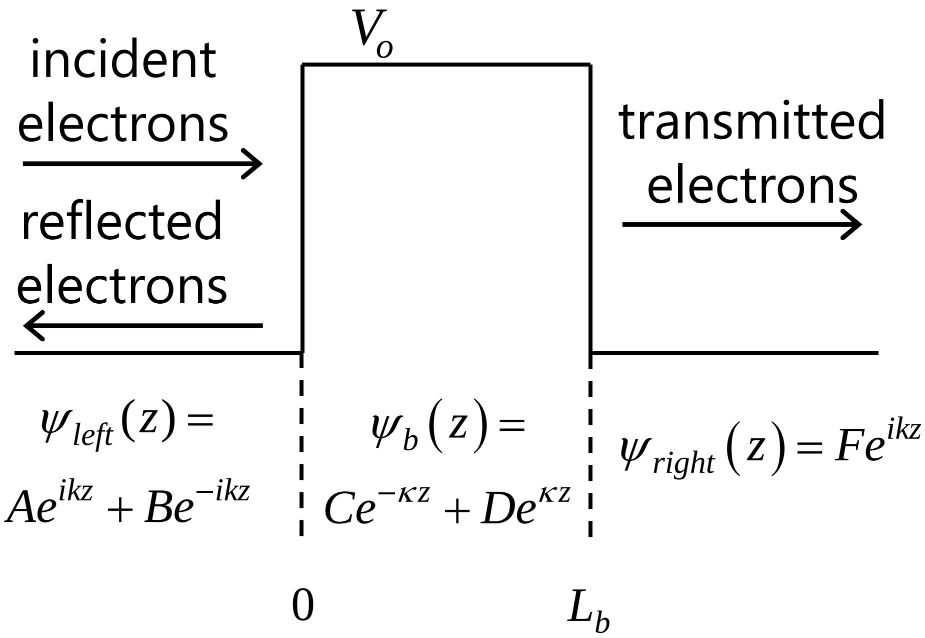
formally, in probability density

that is transmitted through the barrier

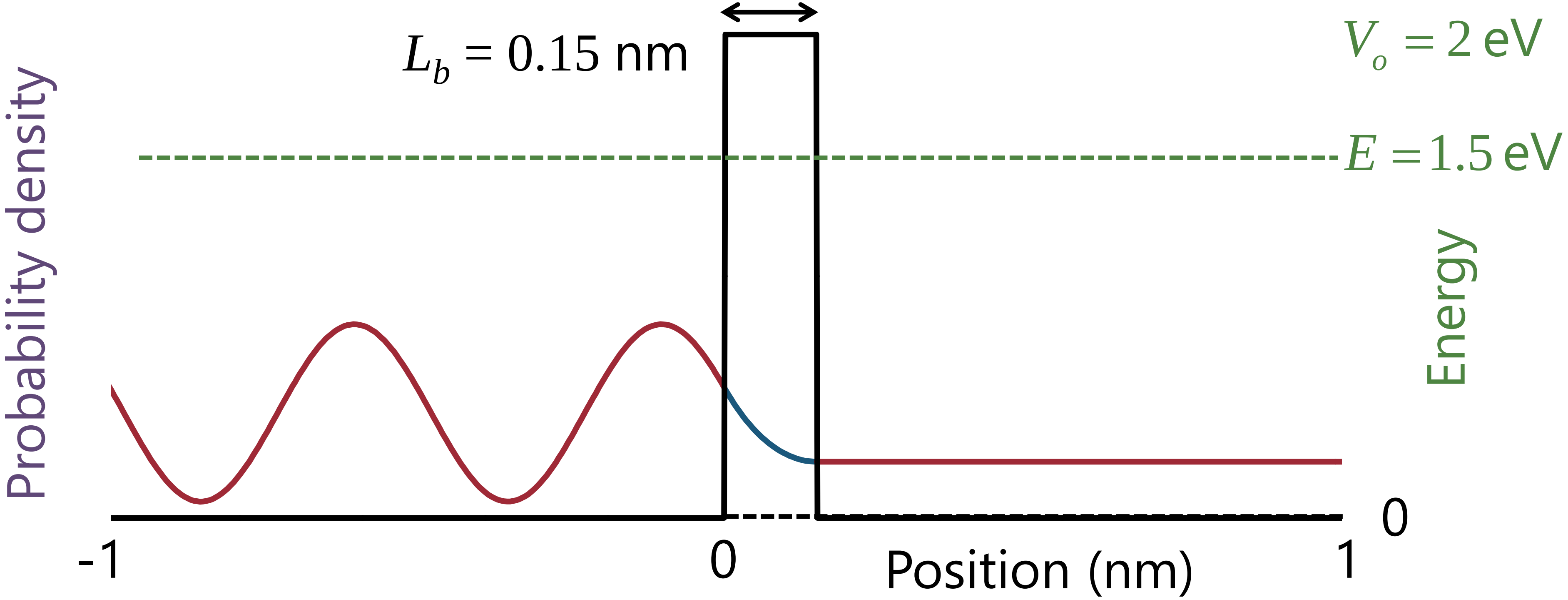
will be the ratio $\frac{|F|^2}{|A|^2}$

We can call this

the current "transmission" through the barrier



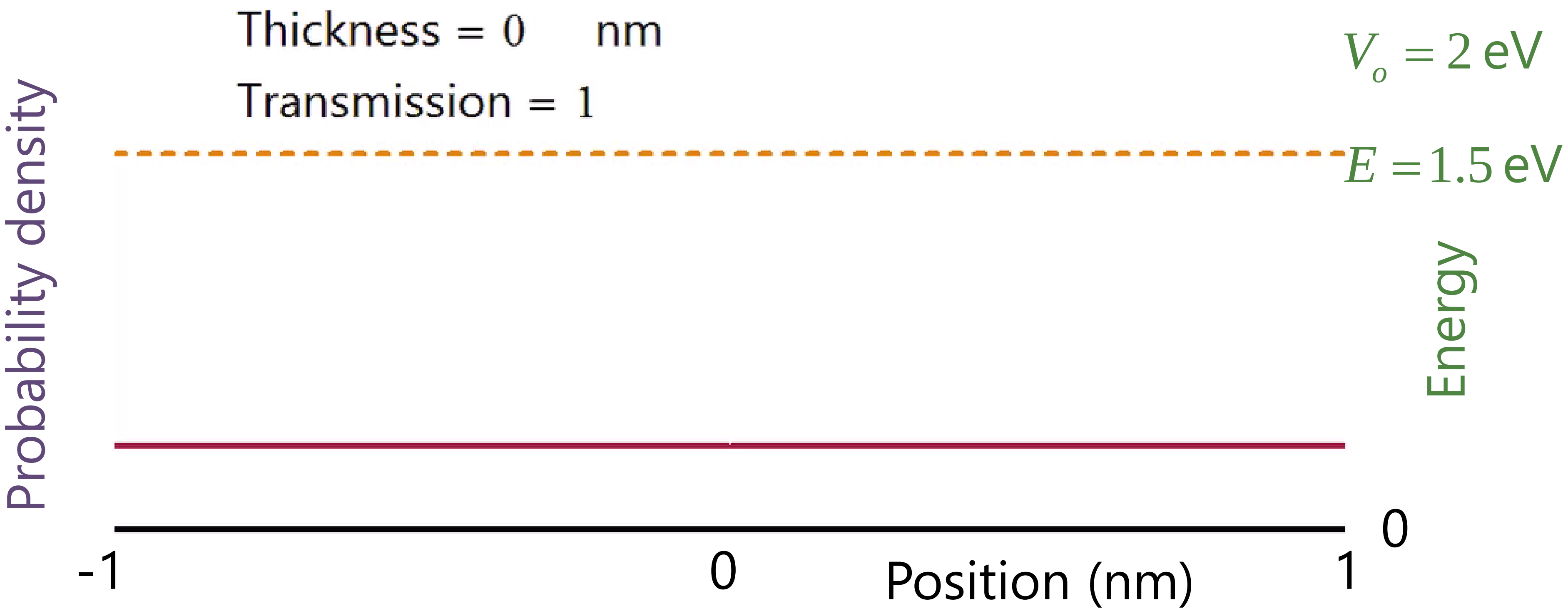
Tunneling through a barrier



Note the weaker standing wave on the left

and the transmission to the right

Tunneling through a barrier



Note the weaker standing wave on the left

and the transmission to the right